Math 125B
Vogler

Iterated Integrals

Defn Let $f: \mathbb{R}^n \to \mathbb{R}$ be defined for all $x_j \in [a,b]$ and $x_k \in [c,d]$ where $j \neq k$ and $\vec{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$. The <u>iterated integral</u> is

$$\int_c^d \int_a^b f(\vec{x})\, dx_j\, dx_k := \int_c^d \left(\int_a^b f(\vec{x})\, dx_j \right) dx_k,$$

when both integrals on the right side exist. This is often referred to as a <u>double integral</u>. When we integrate over a third coordinate x_i with $i \neq j \neq k$, this is called a <u>triple integral</u>. Higher-order iterated integrals (up to order n) can be defined in a similar way.

Lemma (12.30)
Let $R := [a,b] \times [c,d]$ and $f: R \to \mathbb{R}$ be bounded. If $f(x, \cdot)$ is integrable on $[c,d]$ $\forall x \in [a,b]$, then

$$\underline{\iint_R} f\, dA \leq \underline{\int_a^b} \left(\underline{\int_c^d} f(x,y)\, dy \right) dx \leq \overline{\int_a^b} \left(\overline{\int_c^d} f(x,y)\, dy \right) dx \leq \overline{\iint_R} f\, dA.$$

Thm (12.31) <u>Fubini's Theorem</u>
Let $R := [a,b] \times [c,d]$ and let $f: R \to \mathbb{R}$. If $f(x, \cdot) \in \mathcal{R}([c,d])$ $\forall x \in [a,b]$, $f(\cdot, y) \in \mathcal{R}([a,b])$ $\forall y \in [c,d]$, and $f \in \mathcal{R}(R)$, then

$$\iint_R f\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy.$$

-Notes: 1) The hypotheses hold when $f \in C(R)$.
2) This is the fact that allowed you to switch the order of integration in Vector Calculus (mat 21D). Also, it enabled you to compute double integrals as iterated integrals.

Lemma (12.36)

Let $R = [a_1, b_1] \times \dots \times [a_n, b_n]$ and $f: R \to \mathbb{R}$ where $f \in \mathcal{R}(R)$. If $\forall \vec{x} := (x_1, \dots, x_{n-1}) \in R^* := [a_1, b_1] \times \dots \times [a_{n-1}, b_{n-1}]$, $f(\vec{x}, \cdot) \in \mathcal{R}([a_n, b_n])$, then

$$\int_{a_n}^{b_n} f(\vec{x}, t)\,dt$$

is integrable on R^*, and

$$\int_R f(\vec{x}, t)\,d(\vec{x}, t) = \int_{R^*} \int_{a_n}^{b_n} f(\vec{x}, t)\,dt\,d\vec{x}.$$

-Note: This is the fact that enabled you to compute triple integrals by using 3 iterated integrals in Vector Calculus.

Thm (12.39)

Let E be a projectable region in $\mathbb{R}^n$ generated by j, H, ϕ, and ψ. Then E is a Jordan region in $\mathbb{R}^n$. Moreover, if $f: E \to \mathbb{R}$ is continuous on E, then

$$\int_E f(\vec{x})\,d\vec{x} = \int_H \left(\int_{\phi(x_1, \dots, \hat{x}_j, \dots, x_n)}^{\psi(x_1, \dots, \hat{x}_j, \dots, x_n)} f(\vec{x})\,dx_j \right) d(x_1, \dots, \hat{x}_j, \dots, x_n).$$

-Notes: 1) The (many) definitions to understanding exactly what this theorem states are in the book are omitted here for space and time.

2) This fact allowed you to 'break up' a triple integral into a 2D projection, where you can build a double integral, and a 1D height function, where you would bound by a top surface and a bottom one, in Vector Calculus.