Math 125B
Vogler

Jorden Regions

Notation: Unless otherwise stated, $R := [a_1, b_1] \times \dots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is an n-dimensional rectangle.

Defn 1) A grid on R is a collection of n-dimensional rectangles $\mathcal{G} := \{R_1, \dots, R_p\}$ obtained by subdividing each side of R. More specifically, $\forall j = 1, \dots, n$, $\exists n_j \in \mathbb{N}$ and partitions of $[a_j, b_j]$ defined as $P_j = P_j(\mathcal{G}) := \{x_{j,k} : k = 1, \dots, n_j\}$ such that $\mathcal{G}$ is the collection of rectangles of the form $I_1 \times \dots \times I_n$ with $I_j := [x_{j,k-1}, x_{j,k}]$ for $j = 1, \dots, n$ and $k = 1, \dots, n_j$.

Define $\mathcal{P}(R)$ to be the set of all grids on R.

2) A refinement of a grid $\mathcal{G} \in \mathcal{P}(R)$ is a grid $\mathcal{H} \in \mathcal{P}(R)$ where each partition $P_j(\mathcal{H})$ is finer than the corresponding partition $P_j(\mathcal{G})$ (i.e. $P_j(\mathcal{G}) \subseteq P_j(\mathcal{H})$ $\forall j = 1, \dots, n$), and we say $\mathcal{H}$ is finer than $\mathcal{G}$ or $\mathcal{G}$ is coarser than $\mathcal{H}$.

3) The volume of R is $|R| := (b_1 - a_1) \cdot \dots \cdot (b_n - a_n)$, where this is called length when $n=1$ and area when $n=2$.

4) For $E \subseteq \mathbb{R}^n$, let R be rectangle where $E \subseteq R$. Then, we define the outer sums of E with respect to grid $\mathcal{G} \in \mathcal{P}(R)$ to be

$$V(E; \mathcal{G}) := \sum_{R_j \cap E \neq \emptyset} |R_j|,$$

where the empty sum is defined to be zero.

-Notes: a) Let $E \subseteq R$ and $\mathcal{G}, \mathcal{H} \in \mathcal{P}(R)$. If $\mathcal{G}$ is finer than $\mathcal{H}$, then

$$V(E; \mathcal{G}) \leq V(E; \mathcal{H}).$$

b) If $A \subseteq R$, $B \subseteq R$, and $A \subseteq B$, then $V(A; \mathcal{G}) \leq V(B; \mathcal{G})$ $\forall \mathcal{G} \in \mathcal{P}(R)$.

Defn A set $E \subseteq \mathbb{R}^n$ has volume zero if $\forall \varepsilon > 0$ there is a rectangle $R \supseteq E$ and a grid $G \in \mathcal{P}(R)$ where $V(E;G) < \varepsilon$.

Thm (12.4) For all $E \subseteq \mathbb{R}^n$, the following are equivalent:

1) E has volume zero.
2) There exists constant $C > 0$ where $\forall \varepsilon > 0, \exists R \supseteq E$ and $G \in \mathcal{P}(R)$ such that $V(E;G) < C\varepsilon$.
3) $\forall \varepsilon > 0$ there exists a finite collection of cubes Q_k of the same size such that $E \subset \bigcup_{k=1}^{q} Q_k$ and $\sum_{k=1}^{q} |Q_k| < \varepsilon$.

Defn $E \subseteq \mathbb{R}^n$ is a Jorden region if $E \subseteq R$ for some rectangle R and ∂E has volume zero. In this case, we define the volume of E by

$$Vol(E) := \inf_{G \in \mathcal{P}(R)} V(E;G).$$

Note: R is a Jordon Region with $Vol(R) = |R|$.

Thm (12.7) If E_1 and E_2 are Jordon regions, then $E_1 \cup E_2$ is also a Jordon region and $Vol(E_1 \cup E_2) \leq Vol(E_1) + Vol(E_2)$.

Defn Let $\mathcal{E} := \{E_\ell\}_{\ell \in \mathbb{N}}$ be collection of subsets of $\mathbb{R}^n$.

1) $\mathcal{E}$ is nonoverlapping if $\forall j \neq k$ $E_j \cap E_k$ has volume zero.
2) $\mathcal{E}$ is pairwise disjoint if $E_j \cap E_k = \emptyset$ $\forall j \neq k$.

Note: Every collection that is pairwise disjoint is nonoverlapping.

Lemma (12.9) Let $V \subseteq \mathbb{R}^n$ be open and bounded, and let $\vec{\phi}: V \to \mathbb{R}^n$ be 1-1 and $C'(V)$ with $\Delta_{\vec{\phi}} \neq 0$. If E has volume zero and $\overline{E} \subseteq V$, then $\vec{\phi}(E)$ has volume zero.

Thm (12.10) Let $V \subseteq \mathbb{R}^n$ be open and bounded and let $\vec{\phi}: V \to \mathbb{R}^n$ be 1-1 and $C'(V)$ with $\Delta_{\vec{\phi}} \neq 0$. If E is a Jordan region and $\overline{E} \subseteq V$, then $\vec{\phi}(E)$ is a Jordan region.