Notation: Unless otherwise stated, \( R := [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n \) is an \( n \)-dimensional rectangle.

**Definition** A grid on \( R \) is a collection of \( n \)-dimensional rectangles \( G := \{ R_1, \ldots, R_p \} \) obtained by subdividing each side of \( R \). More specifically, \( \forall j = 1, \ldots, n, \exists n_j \in \mathbb{N} \) and partitions of \([a_j, b_j]\) defined as \( p_j = p_j(G) := \{ x_{j,k} : k = 1, \ldots, n_j \} \) such that \( G \) is the collection of rectangles of the form \( I_1 \times \ldots \times I_n \) with \( I_j := [x_{j,k-1}, x_{j,k}] \) for \( j = 1, \ldots, n \) and \( k = 1, \ldots, n_j \).

Define \( \mathcal{P}(R) \) to be the set of all grids on \( R \).

2) A refinement of a grid \( G \in \mathcal{P}(R) \) is a grid \( \mathcal{H} \in \mathcal{P}(R) \) where each partition \( p_j(G) \) is finer than the corresponding partition \( p_j(\mathcal{H}) \) (i.e. \( p_j(G) \subseteq p_j(\mathcal{H}) \) \( \forall j = 1, \ldots, n \)), and we say \( \mathcal{H} \) is finer than \( G \) or \( G \) is coarser than \( \mathcal{H} \).

3) The volume of \( R \) is \( \text{vol}(R) := (b_1 - a_1) \ldots (b_n - a_n) \), where this is called length when \( n = 1 \) and area when \( n = 2 \).

4) For \( E \subseteq \mathbb{R}^n \), let \( R \) be a rectangle such that \( E \subseteq R \). Then, we define the outer sums of \( E \) with respect to grid \( G \in \mathcal{P}(R) \) to be

\[
\text{vol}(E; G) := \sum_{R_j \cap E \neq \emptyset} \text{vol}(R_j),
\]

where the empty sum is defined to be zero.

**Notes:**

a) Let \( E \subseteq R \) and \( G, \mathcal{H} \in \mathcal{P}(R) \). If \( G \) is finer than \( \mathcal{H} \), then

\[
\text{vol}(E; G) \leq \text{vol}(E; \mathcal{H}).
\]

b) If \( A \subseteq R, B \subseteq R, \) and \( A \subseteq B, \) then \( \text{vol}(A; G) \leq \text{vol}(B; G) \) \( \forall G \in \mathcal{P}(R) \).
Defn A set $E \subseteq \mathbb{R}^n$ has **volume zero** if $\forall \varepsilon > 0$ there is a rectangle $R \supseteq E$ and a grid $G \in \mathcal{P}(R)$ where $V(E;G) < \varepsilon$.

Thm (12.9) For all $E \subseteq \mathbb{R}^n$ the following are equivalent:

1) $E$ has volume zero.
2) There exists constant $c > 0$ where $\forall \varepsilon > 0, \exists R \supseteq E$ and $G \in \mathcal{P}(R)$ such that $V(E;G) < c\varepsilon$.
3) $\forall \varepsilon > 0$ there exists a finite collection of cubes $Q_k$ of the same size such that $E \subseteq \bigcup_{k=1}^q Q_k$ and $\sum_{k=1}^q |Q_k| < \varepsilon$.

Defn $E \subseteq \mathbb{R}^n$ is a **Jordan region** if $E \subseteq R$ for some rectangle $R$ and $\partial E$ has volume zero. In this case, we define the volume of $E$ by

$$\text{Vol}(E) := \inf_{G \in \mathcal{P}(R)} V(E;G).$$

Note: $R$ is a Jordan Region with $\text{Vol}(R) = |R|$. Thm (12.7) If $E_1$ and $E_2$ are Jordan regions, then $E_1 \cup E_2$ is also a Jordan region and $\text{Vol}(E_1 \cup E_2) \leq \text{Vol}(E_1) + \text{Vol}(E_2)$.

Defn Let $E := \{E_1 \subseteq \mathbb{R}^n \}$ be collection of subsets of $\mathbb{R}^n$.

1) $E$ is **nonoverlapping** if $\forall j \neq k$ $E_j \cap E_k$ has volume zero.
2) $E$ is **pairwise disjoint** if $E_j \cap E_k = \emptyset$ $\forall j \neq k$.

Note: Every collection that is pairwise disjoint is nonoverlapping.

Lemma (12.9) Let $V \subseteq \mathbb{R}^n$ be open and bounded, and let $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ be $1$-$1$ and $\phi'(V)$ with $\Delta \tilde{\phi} \neq 0$. If $E$ has volume zero and $E \subseteq V$, then $\phi(E)$ has volume zero.

Thm (12.10) Let $V \subseteq \mathbb{R}^n$ be open and bounded, and let $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ be $1$-$1$ and $\phi'(V)$ with $\Delta \tilde{\phi} \neq 0$. If $E$ is a Jordan region and $E \subseteq V$, then $\phi(E)$ is a Jordan region.