Math 125B
Vogler

$\mathbb{R}^n$ and Linear Transformations

Defn Let $\vec{x}=(x_1,x_2,\cdots,x_n)$, $\vec{y}=(y_1,\cdots,y_n)\in\mathbb{R}^n$ and $\alpha\in\mathbb{R}$.

1) The sum of vectors $\vec{x}$ and $\vec{y}$ is the vector
$$\vec{x}+\vec{y}:=(x_1+y_1,x_2+y_2,\cdots,x_n+y_n).$$
2) The difference of $\vec{x}$ and $\vec{y}$ is
$$\vec{x}-\vec{y}:=(x_1-y_1,x_2-y_2,\ldots,x_n-y_n).$$
3) The product of scalar α and the vector $\vec{x}$ is the vector
$$\alpha\vec{x}:=(\alpha x_1,\alpha x_2,\ldots,\alpha x_n)$$
4) The dot product of the vectors $\vec{x}$ and $\vec{y}$ is the scalar
$$\vec{x}\cdot\vec{y}:=\sum_{i=1}^n x_i y_i = x_1y_1+x_2y_2+\cdots+x_ny_n.$$
5) The norm (or magnitude) of $\vec{x}$ is the scalar
$$\|\vec{x}\|:=\sqrt{\sum_{i=1}^n (x_i)^2}=\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$$
6) The ℓ^1-norm of $\vec{x}$ is the scalar $\|\vec{x}\|_1:=\sum_{i=1}^n |x_i|$
7) The sup-norm of $\vec{x}$ is the scalar $\|\vec{x}\|_\infty:=\max_{i=1,\ldots,n}\{|x_i|\}$
8) The distance between two points $\vec{x}$ and $\vec{y}$ is the scalar
$$\text{dist}(\vec{x},\vec{y}):=\|\vec{x}-\vec{y}\|.$$

-Note: Vector addition 1) and products 3) & 4) follows basic rules, analogous to their corresponding rules in $\mathbb{R}$, which are listed in Thm 8.2.

Defn Let $\vec{x},\vec{y}\in\mathbb{R}^n$ be non zero vectors.

1) $\vec{x}$ and $\vec{y}$ are parallel ($\parallel$) if $\exists t\in\mathbb{R}$ such that $\vec{x}=t\vec{y}$.
2) $\vec{x}$ and $\vec{y}$ are orthogonal ($\perp$) if $\vec{x}\cdot\vec{y}=0$.

Thm (8.5) Cauchy-Schwarz Inequality $|\vec{x}\cdot\vec{y}|\le\|\vec{x}\|\,\|\vec{y}\|$ $\forall\vec{x},\vec{y}\in\mathbb{R}^n$.

Thm (8.6) Let $\vec{x},\vec{y}\in\mathbb{R}^n$.

1) $\|\vec{x}\|\geq 0$ and $\|\vec{x}\|=0$ if and only if $\vec{x}=0$.
2) $\|\alpha\vec{x}\| = |\alpha|\,\|\vec{x}\|$ $\forall\,\alpha\in\mathbb{R}$
3) $\|\vec{x}+\vec{y}\|\leq\|\vec{x}\|+\|\vec{y}\|$ and $\|\vec{x}-\vec{y}\|\geq\|\vec{x}\|-\|\vec{y}\|$. (Triangle Inequalities)

- Note: This and the definition of $dist(\vec{x},\vec{y})=:d(\vec{x},\vec{y})$ make $(\mathbb{R}^n,d)$ a metric space as discussed in Math 25 and 125A.

Thm (8.7) Let $\vec{x}\in\mathbb{R}^n$.

1) $\|\vec{x}\|_\infty\leq\|\vec{x}\|\leq\sqrt{n}\,\|\vec{x}\|_\infty$ and 2) $\|\vec{x}\|\leq\|\vec{x}\|_1\leq\sqrt{n}\,\|\vec{x}\|$

Defn The <u>cross product</u> of $\vec{x}=(x_1,x_2,x_3)$, $\vec{y}=(y_1,y_2,y_3)\in\mathbb{R}^3$ is the vector $\vec{x}\times\vec{y}:=(x_2y_3-x_3y_2,\ x_3y_1-x_1y_3,\ x_1y_2-x_2y_1)$.

-Note: Properties of this product can be found in Thm 8.9.

Defn A function $T:\mathbb{R}^n\to\mathbb{R}^m$ is <u>linear</u> if it satisfies

1) $T(\vec{x}+\vec{y})=T(\vec{x})+T(\vec{y})$ and 2) $T(\alpha\vec{x})=\alpha T(\vec{x})$ $\forall\,\vec{x},\vec{y}\in\mathbb{R}^n$, $\forall\,\alpha\in\mathbb{R}$.

<u>Notation:</u> $\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ is the set of all linear functions from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Thm (8.15) For each $T\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$ there is a matrix $B=[b_{ij}]_{m\times n}$ such that $T(\vec{x})=B\vec{x}$ $\forall\,\vec{x}\in\mathbb{R}^n$. Also, this matrix is unique and can be computed by

$$(b_{1j},b_{2j},\dots,b_{mj})=T(\hat{e}_j)\quad\forall\, j=1,\dots,n,$$

where $\hat{e}_1,\hat{e}_2,\dots,\hat{e}_n$ is the usual basis in $\mathbb{R}^n$.

Note: $B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & \cdots & & b_{2n} \\ \vdots & & & \vdots \\ b_{m1} & \cdots & & b_{mn}\end{bmatrix}$

Defn Let $T\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^m)$. The <u>operator norm</u> is the extended real number

$$\|T\|:=\sup_{\|\vec{x}\|\neq 0}\frac{\|T(\vec{x})\|}{\|\vec{x}\|}.$$

Thm (8.17) If $T\in\mathcal{L}(\mathbb{R}^n$

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