Math 125B
Vogler

Properties of Partial Derivatives and Integrals

Thm (11.2)

Let $V \subseteq \mathbb{R}^2$ be open, $(x_0, y_0) \in \mathbb{R}^2$, and $f: V \to \mathbb{R}$. If $f \in \mathcal{C}^1(V)$ and if one of the mixed second partial derivatives of f exists on V and is continuous at the point (x_0, y_0), then the other partial derivative exists at (x_0, y_0) and

$$f_{xy}(x_0, y_0) = f_{yx}(y_0, x_0).$$

- Note: The conditions are met if $f \in \mathcal{C}^2(V)$.

Thm (11.4)

Let $H = [a,b] \times [c,d]$ and $f: H \to \mathbb{R}$ be continuous. If

$$F(y) := \int_a^b f(x,y)\,dx,$$

then F is continuous on $[c,d]$. More specifically, we have

$$\lim_{\substack{y \to y_0 \\ y \in [c,d]}} \int_a^b f(x,y)\,dx = \int_a^b \lim_{\substack{y \to y_0 \\ y \in [c,d]}} f(x,y)\,dx \quad \forall y_0 \in [c,d].$$

Thm (11.5)

Let $H = [a,b] \times [c,d]$ and $f: H \to \mathbb{R}$. Let $f(\cdot, y) \in \mathcal{R}([a,b])$, $\forall y \in [c,d]$ and $f_y(x,\cdot)$ exists on $[c,d]$, $\forall x \in [a,b]$. If $f_y(x,y) \in \mathcal{C}(H)$, then

$$\frac{d}{dy} \int_a^b f(x,y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx \quad \forall y \in [c,d].$$

- Note: The conditions are satisfied if $f \in \mathcal{C}^1(H)$.

- Note: This list is restricted to real-valued functions of two variables and can easily be extended to vector-valued functions of more than two variables (Optional HW).