Defn

1) The **Cartesian product** of the collection of sets \( E_1, E_2, \ldots, E_n \) is the set of \( n \)-tuples (or \( n \) vectors) defined as:

\[
E_1 \times E_2 \times \cdots \times E_n := \{(x_1, x_2, \ldots, x_n) : x_j \in E_j \text{ for } j = 1, 2, \ldots, n \}
\]

**Note:** The Cartesian product of \( n \) subsets of \( \mathbb{R} \) is a subset of \( \mathbb{R}^n \).

2) An **\( n \)-dimensional rectangle** is the Cartesian product of \( n \) closed, nondegenerate, bounded intervals like

\[
[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n].
\]

An **\( n \)-dimensional cube** is an \( n \)-dimensional rectangle where all the sides have equal length (i.e. \( \exists \ell \in \mathbb{R} \) such that \( b_j - a_j = \ell \) \( \forall j = 1, \ldots, n \)).

3) Let \( f : \mathbb{R}^{k} \times \mathbb{R}^{k} \times \cdots \times \mathbb{R}^{k} \rightarrow \mathbb{R} \). We define the corresponding function \( g : [a, b] \rightarrow \mathbb{R} \) by

\[
g(t) := f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n) \quad \forall t \in [a, b].
\]

a) The **partial integral** of \( f \) on \([a, b]\) with respect to \( x_i \)

\[
\int_a^b f(x_1, \ldots, x_n) \, dx_i := \int_a^b g(t) \, dt.
\]

b) If \( g \) is differentiable at \( t_0 \in (a, b) \), then the **partial derivative** of \( f \) at \((x_1, \ldots, x_{i-1}, t_0, x_{i+1}, \ldots, x_n) \) with respect to \( x_i \)

\[
\frac{\partial f}{\partial x_i} (x_1, \ldots, x_{i-1}, t_0, x_{i+1}, \ldots, x_n) := g'(t_0).
\]

Equivallently, the partial derivative \( f_{x_i} \) exists at a point \( \hat{a} \in \mathbb{R}^n \) if the limit \( \lim_{h \to 0} \frac{f(\hat{a} + h\hat{e}_j) - f(\hat{a})}{h} \) exists.
Note: The notions of partial derivatives and integrals by restricting functions of many variables down to functions of one variable, so any result on functions of one variable apply to partial derivatives and integrals like FTOC, Product Rule, Mean Value Theorems, etc.

4) For vector-valued functions \( \vec{F} = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \rightarrow \mathbb{R}^m \), if all the first-order partial derivatives \( \frac{\partial f_k}{\partial x_j} (\vec{a}) \) \( \forall k = 1, \ldots, m \), for a specific \( j = 1, \ldots, n \), then we define the first-order partial derivative of \( \vec{F} \) with respect to \( x_j \) to be

\[
\vec{F}_{x_j} (\vec{a}) := \frac{\partial \vec{F}}{\partial x_j} (\vec{a}) := \left( \frac{\partial f_1}{\partial x_j} (\vec{a}), \ldots, \frac{\partial f_m}{\partial x_j} (\vec{a}) \right) \quad \text{for} \quad \vec{a} \in \mathbb{R}^n.
\]

5) Higher order partial derivatives are defined by iteration. For example, the second-order partial derivative of \( \vec{F} \) with respect to \( x_j \) and \( x_k \) is

\[
\vec{F}_{x_j x_k} := \frac{\partial^2 \vec{F}}{\partial x_k \partial x_j} := \frac{\partial}{\partial x_k} \left( \frac{\partial \vec{F}}{\partial x_j} \right)
\]

when it exists. These are called mixed when \( j \neq k \).

**Defn** Let \( V \subseteq \mathbb{R}^n \) be open, \( \vec{F} : V \rightarrow \mathbb{R}^m \) and \( p \in \mathbb{N} \).

1) \( C^p(V) \) is the set of functions \( \vec{F} \) where each \( k \)th-order partial derivative exists and is continuous \( \forall k \in \mathbb{N} \) with \( k \leq p \).

2) \( C^\infty(V) \) is the set of functions \( \vec{F} \) where \( \vec{F} \in C^p(V) \) \( \forall p \in \mathbb{N} \).

- **Notes:**
  a) \( C^\infty(V) \subseteq C^p(V) \subseteq C^q(V) \) \( \forall p, q \in \mathbb{N} \) with \( q \leq p \).
  b) We say \( \vec{F} \) is smooth if \( \vec{F} \in C^\infty(V) \).

**Notation:** Unless otherwise stated, \( H := [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n \) is an \( n \)-dimensional rectangle.