Math 125B
Vogler

Partial Derivatives and Integrals

Defn

1) The Cartesian product of the collection of sets $E_1, E_2, \dots, E_n$ is the set of n-tuples (or n vectors) defined as

$$E_1 \times E_2 \times \dots \times E_n := \{(x_1, x_2, \dots, x_n) : x_j \in E_j \text{ for } j = 1, 2, \dots, n\}.$$

- Note: The Cartesian product of n subsets of $\mathbb{R}$ is a subset of $\mathbb{R}^n$.

2) An n-dimensional rectangle is the Cartesian product of n closed, nondegenerate, bounded intervals like $[a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$. An n-dimensional cube is an n-dimensional rectangle where all the sides have equal length (i.e. $\exists\, \ell \in \mathbb{R}$ such that $|b_j - a_j| = \ell \;\; \forall j = 1, \dots, n$).

3) Let $f : \{x_1\} \times \dots \times \{x_{j-1}\} \times [a,b] \times \{x_{j+1}\} \times \dots \times \{x_n\} \to \mathbb{R}$. We define the corresponding function $g : [a,b] \to \mathbb{R}$ by

$$g(t) := f(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n) \quad \forall t \in [a,b].$$

a) The partial integral of f on $[a,b]$ with respect to x_j is

$$\int_a^b f(x_1, \dots, x_n)\, dx_j := \int_a^b g(t)\, dt.$$

b) If g is differentiable at $t_0 \in (a,b)$, then the partial derivative of f at $(x_1, \dots, x_{j-1}, t_0, x_{j+1}, \dots, x_n)$ with respect to x_j is

$$\frac{\partial f}{\partial x_j}(x_1, \dots, x_{j-1}, t_0, x_{j+1}, \dots, x_n) := g'(t_0).$$

Equivallently, the partial derivative f_{x_j} exists at a point $\vec{a} \in \mathbb{R}^n$ if the limit $\frac{\partial f}{\partial x_j}(\vec{a}) := \lim_{h \to 0} \frac{f(\vec{a} + h\hat{e}_j) - f(\vec{a})}{h}$ exists.

-Note: The notions of partial derivatives and integrals by restricting functions of many variables down to functions of one variable, so any result on functions of one variable apply to partial derivatives and integrals like FTOC, Product Rule, Mean Value Theorems, etc.

4) For vector-valued functions $\vec{f}=(f_1, f_2, \ldots, f_m): \mathbb{R}^n \to \mathbb{R}^m$, if all the first-order partial derivatives $\frac{\partial f_k}{\partial x_j}(\vec{a})$ $\forall k=1,\ldots,m$, for a specific $j=1,\ldots,n$, then we define the <u>first-order partial derivative</u> of $\vec{f}$ with respect to x_j to be

$$\vec{f}_{x_j}(\vec{a}) := \frac{\partial \vec{f}}{\partial x_j}(\vec{a}) := \left(\frac{\partial f_1}{\partial x_j}(\vec{a}), \ldots, \frac{\partial f_m}{\partial x_j}(\vec{a})\right) \quad \text{for } \vec{a} \in \mathbb{R}^n.$$

5) Higher order partial derivatives are defined by iteration. For example, the <u>second-order partial derivative</u> of $\vec{f}$ with respect to x_j and x_k is

$$\vec{f}_{x_j x_k} := \frac{\partial^2 \vec{f}}{\partial x_k \partial x_j} := \frac{\partial}{\partial x_k}\left(\frac{\partial \vec{f}}{\partial x_j}\right)$$

when it exists. These are called <u>mixed</u> when $j \neq k$.

<u>Defn</u> Let $V \subseteq \mathbb{R}^n$ be open, $\vec{f}: V \to \mathbb{R}^m$, and $p \in \mathbb{N}$.

1) $C^p(V)$ is the set of functions $\vec{f}$ where each kth-order partial derivative exists and is continuous $\forall k \in \mathbb{N}$ with $k \leq p$.

2) $C^\infty(V)$ is the set of functions $\vec{f}$ where $\vec{f} \in C^p(V)$ $\forall p \in \mathbb{N}$.

-Notes: a) $C^\infty(V) \subseteq C^p(V) \subseteq C^q(V)$ $\forall p, q \in \mathbb{N}$ with $q \leq p$.

b) We say $\vec{f}$ is <u>smooth</u> if $\vec{f} \in C^\infty(V)$.

<u>Notation</u>: Unless otherwise stated, $H := [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n$ is an n-dimensional rectangle.