Math 125B
Vogler

Taylor's Formula

Detn Let $p \in \mathbb{N}$ with $p \geq 1$, $V \subseteq \mathbb{R}^n$ be open, $\vec{a} \in V$, and $f: V \to \mathbb{R}$. Also, assume all the pth partial derivatives exist at $\vec{a}$. We define the *pth-order total differential of f at $\vec{a}$* is

$$D^{(p)} f(\vec{a}; h) := \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^p f}{\partial x_{i_1} \cdots \partial x_{i_p}}(\vec{a})\, h_{i_1} \cdots h_{i_p}, \quad \vec{h} = (h_1, h_2, \dots, h_n) \in \mathbb{R}^n.$$

-Notes: 1) For $p = 1$, $D^{(1)} f(\vec{a}; \Delta\vec{x}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(\vec{a}) \Delta x_j = \nabla f(\vec{a}) \cdot \Delta\vec{x} = df$, which is the differential we studied in Section 11.3.

2) You can build the next (pth) total differential by using the previous one ($(p-1)$th) with the following formula

$$D^{(p)} f(\vec{a}; \vec{h}) = D^{(1)}\left[D^{(p-1)} f\right](\vec{a}; \vec{h})$$
$$= \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\sum_{i_1=1}^{n} \cdots \sum_{i_{p-1}=1}^{n} \frac{\partial^{p-1} f}{\partial x_{i_1} \cdots \partial x_{i_{p-1}}}(\vec{a})\, h_{i_1} \cdots h_{i_{p-1}} \right) h_j$$

for $p > 1$.

Thm (11.37) Taylor's Formula on $\mathbb{R}^n$

Let $p \in \mathbb{N}$, $V \subseteq \mathbb{R}^n$ be open, $\vec{x}, \vec{a} \in V$, and $f: V \to \mathbb{R}$. If the pth total differential of f exists on V and $L(\vec{x}; \vec{a}) \subseteq V$, then $\exists\, c \in L(\vec{x}; \vec{a})$ such that

$$f(\vec{x}) = f(\vec{a}) + \sum_{k=1}^{p-1} \frac{1}{k!} D^{(k)} f(\vec{a}; \vec{h}) + \frac{1}{p!} D^{(p)} f(\vec{c}; \vec{h}) \quad (*)$$

for $\vec{h} := \vec{x} - \vec{a}$.

-Note: The first two terms on the RHS of $(*)$ is sometimes referred to as the $(p-1)$th-order Taylor Polynomial while the last term is called the $(p-1)$th-order remainder.