Defn Let \( p \in \mathbb{N} \) with \( p \geq 1 \), \( V \subseteq \mathbb{R}^n \) be open, \( \vec{a} \in V \), and \( f : V \to \mathbb{R} \). Also, assume all the \( p \)th partial derivatives exist at \( \vec{a} \). We define the \( p \)th-order total differential of \( f \) at \( \vec{a} \) as

\[
D^{(p)} f(\vec{a}; \vec{h}) := \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \frac{\partial^p f}{\partial x_{i_1} \cdots \partial x_{i_p}} (\vec{a}) \ h_{i_1} \cdots h_{i_p}, \quad \vec{h} = (h_1, h_2, \ldots, h_n) \in \mathbb{R}^n.
\]

Notes: 1) For \( p = 1 \), \( D^{(1)} f(\vec{a}, \Delta \vec{x}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} (\vec{a}) \Delta x_j = \nabla f(\vec{a}) \cdot \Delta \vec{x} = df \), which is the differential we studied in Section 11.3.
2) You can build the next \( (p+1) \)th total differential by using the previous one \((p-1)\)th with the following formula

\[
D^{(p)} f(\vec{a}; \vec{h}) = D^{(1)} \left[ D^{(p-1)} f \right](\vec{a}; \vec{h}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( \sum_{i_1=1}^{n} \cdots \sum_{i_{p-1}=1}^{n} \frac{\partial^{p-1} f}{\partial x_{i_1} \cdots \partial x_{i_{p-1}}} (\vec{a}) \ h_{i_1} \cdots h_{i_{p-1}} \right) h_j
\]

for \( p > 1 \).

Thm (11.37) Taylor's Formula on \( \mathbb{R}^n \)
Let \( p \in \mathbb{N} \), \( V \subseteq \mathbb{R}^n \) be open, \( \vec{x}, \vec{a} \in V \), and \( f : V \to \mathbb{R} \). If the \( p \)th total differential of \( f \) exists on \( V \) and \( L(\vec{x}; \vec{a}) \subseteq V \), then \( \exists \vec{c} \in L(\vec{x}; \vec{a}) \) such that

\[
f(\vec{x}) = f(\vec{a}) + \sum_{k=1}^{p-1} \frac{1}{k!} D^{(k)} f(\vec{a}; \vec{h}) + \frac{1}{p!} D^{(p)} f(\vec{c}; \vec{h}) \quad (*)
\]

for \( \vec{h} := \vec{x} - \vec{a} \).

Note: The first two terms on the RHS of (*) is sometimes referred to as the \((p-1)\)th-order Taylor Polynomial while the last term is called the \((p-1)\)th-order remainder.