for all $y \in [c, d]$ which satisfy $|y - y_0| < \delta$. Then

$$\begin{aligned}|F(y) - F(y_0)| &\leq \left|F(y) - \int_A^B f(x, y)\,dx\right| + \left|\int_A^B (f(x, y) - f(x, y_0))\,dx\right| \\ &\quad + \left|F(y_0) - \int_A^B f(x, y_0)\,dx\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\end{aligned}$$

for all $y \in [c, d]$ which satisfy $|y - y_0| < \delta$. ∎

The proof of Theorem 11.5 can be modified to prove the following result.

***11.9 Theorem.** *Suppose that $a < b$ are extended real numbers, that $c < d$ are finite real numbers, that $f : (a, b) \times [c, d] \to \mathbf{R}$ is continuous, and that the improper integral*

$$F(y) = \int_a^b f(x, y)\,dx$$

exists for all $y \in [c, d]$. If $f_y(x, y)$ exists and is continuous on $(a, b) \times [c, d]$ and if

$$\phi(y) = \int_a^b \frac{\partial f}{\partial y}(x, y)\,dx$$

converges uniformly on $[c, d]$, then F is differentiable on $[c, d]$ and $F'(y) = \phi(y)$; that is,

$$\frac{d}{dy}\int_a^b f(x, y)\,dx = \int_a^b \frac{\partial f}{\partial y}(x, y)\,dx$$

for all $y \in [c, d]$.

For a result about interchanging two partial integrals, see Theorem 12.31 and Exercise 12.3.10.

EXERCISES

11.1.1. Compute all mixed second-order partial derivatives of each of the following functions and verify that the mixed partial derivatives are equal.

a) $f(x, y) = xe^y$ b) $f(x, y) = \cos(xy)$ c) $f(x, y) = \dfrac{x + y}{x^2 + 1}$

11.1.2. For each of the following functions, compute f_x and determine where it is continuous.

a)
$$f(x,y) = \begin{cases} \dfrac{x^4+y^4}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

b)
$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{\sqrt[3]{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

11.1.3. Suppose that $r > 0$, that $\mathbf{a} \in \mathbf{R}^n$, and that $\mathbf{f} : B_r(\mathbf{a}) \to \mathbf{R}^m$. If all first-order partial derivatives of $\mathbf{f}$ exist on $B_r(\mathbf{a})$ and satisfy $\mathbf{f}_{x_j}(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in B_r(\mathbf{a})$ and all $j = 1, 2, \dots, n$, prove that $\mathbf{f}$ has only one value on $B_r(\mathbf{a})$.

11.1.4. Suppose that $H = [a,b] \times [c,d]$ is a rectangle, that $f : H \to \mathbf{R}$ is continuous, and that $g : [a,b] \to \mathbf{R}$ is integrable. Prove that

$$F(y) = \int_a^b g(x) f(x,y)\,dx$$

is uniformly continuous on $[c,d]$.

11.1.5. Evaluate each of the following expressions.

a)
$$\lim_{y\to 0} \int_0^1 e^{x^3y^2+x}\,dx$$

b)
$$\frac{d}{dy}\int_0^1 \sin(e^x y - y^3 + \pi - e^x)\,dx \quad \text{at } y = 1$$

c)
$$\frac{\partial}{\partial x}\int_1^3 \sqrt{x^3+y^3+z^3-2}\,dz \quad \text{at } (x,y) = (1,1)$$

11.1.6. Suppose that f is a continuous real function.

a) If $\int_0^1 f(x)\,dx = 1$, find the exact value of

$$\lim_{y\to 0}\int_0^2 f(|x-1|)e^{x^2y+xy^2}\,dx.$$

b) If f is $\mathcal{C}^1$ on $\mathbf{R}$ and $\int_0^\pi f'(x)\sin x\,dx = e$, find the exact value of

$$e + \lim_{y\to 0}\int_0^\pi f(x)\cos(y^5 + \sqrt[3]{y} + x)\,dx.$$

c) If $\int_0^1 f(\sqrt{x})e^x\,dx = 6$, find the exact value of

$$\frac{d}{dx}\int_0^1 f(y)e^{xy+y^2}\,dy \quad \text{at } x = 0.$$

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***11.1.7.** Evaluate each of the following expressions.

a)
$$\lim_{y\to 0+} \int_0^1 \frac{x\cos y}{\sqrt[3]{1-x+y}}\,dx$$

b)
$$\frac{d}{dy}\int_\pi^\infty \frac{e^{-xy}\sin x}{x}\,dx \qquad \text{at } y=1$$

***11.1.8.** a) Prove that

$$\int_0^1 \frac{\cos(x^2+y^2)}{\sqrt{x}}\,dx$$

converges uniformly on $(-\infty,\infty)$.

b) Prove that $\int_0^\infty e^{-xy}dx$ converges uniformly on $[1,\infty)$.

c) Prove that $\int_0^\infty ye^{-xy}dx$ exists for each $y\in[0,\infty)$ and converges uniformly on any $[a,b]\subset(0,\infty)$ but that it does not converge uniformly on $[0,1]$.

***11.10 Definition.**

The *Laplace transform* of a function $f:(0,\infty)\to\mathbf{R}$ is said to exist at a point $s\in(0,\infty)$ if and only if the integral

$$\mathcal{L}\{f\}(s) := \int_0^\infty e^{-st}f(t)\,dt$$

converges. (*Note*: This integral is improper at ∞ and may be improper at 0.)

***11.1.9.** Prove that

a)
$$\mathcal{L}\{1\}(s)=\frac{1}{s}, \qquad s>0$$

b)
$$\mathcal{L}\{t^n\}(s)=\frac{n!}{s^{n+1}}, \qquad s>0,\ n\in\mathbf{N}$$

c)
$$\mathcal{L}\{e^{at}\}(s)=\frac{1}{s-a}, \qquad s>a,\ a\in\mathbf{R}$$

d)
$$\mathcal{L}\{\cos(bt)\}(s)=\frac{s}{s^2+b^2}, \qquad s>0,\ b\in\mathbf{R}$$

e)
$$\mathcal{L}\{\sin(bt)\}(s)=\frac{b}{s^2+b^2}, \qquad s>0,\ b\in\mathbf{R}$$

***11.1.10.** Suppose that $f : (0, \infty) \to \mathbf{R}$ is continuous and bounded and that $\mathcal{L}\{f\}$ exists at some $a \in (0, \infty)$. Let

$$\phi(t) = \int_0^t e^{-au} f(u)\, du, \qquad t \in (0, \infty).$$

a) Prove that

$$\int_0^N e^{-st} f(t)\, dt = \phi(N) e^{-(s-a)N} + (s-a) \int_0^N e^{-(s-a)t} \phi(t)\, dt$$

for all $N \in \mathbf{N}$.

b) Prove that the integral $\int_0^\infty e^{-(s-a)t}\phi(t)dt$ converges uniformly on $[b, \infty)$ for any $b > a$ and

$$\int_0^\infty e^{-st} f(t)\, dt = (s-a) \int_0^\infty e^{-(s-a)t} \phi(t)\, dt, \qquad s > a.$$

c) Prove that $\mathcal{L}\{f\}$ exists, is continuous on (a, ∞), and satisfies

$$\lim_{s \to \infty} \mathcal{L}\{f\}(s) = 0.$$

d) Let $g(t) = tf(t)$ for $t \in (0, \infty)$. Prove that $\mathcal{L}\{f\}$ is differentiable on (a, ∞) and

$$\frac{d}{ds} \mathcal{L}\{f\}(s) = -\mathcal{L}\{g\}(s)$$

for all $s \in (a, \infty)$.

e) If, in addition, f' is continuous and bounded on $(0, \infty)$, prove that

$$\mathcal{L}(f')(s) = s\mathcal{L}(f)(s) - f(0)$$

for all $s \in (a, \infty)$.

***11.1.11.** Using Exercises 11.1.9 and 11.1.10, find the Laplace transforms of each of the functions te^t, $t \sin \pi t$, and $t^2 \cos t$.

11.2 THE DEFINITION OF DIFFERENTIABILITY

In this section we define what it means for a vector function to be differentiable at a point. Whatever our definition, we expect two things: If $\mathbf{f}$ is differentiable at $\mathbf{a}$, then $\mathbf{f}$ will be continuous at $\mathbf{a}$, and all first-order partial derivatives of $\mathbf{f}$ will exist at $\mathbf{a}$.

Working by analogy with the one-variable case, we guess that $\mathbf{f}$ is differentiable at $\mathbf{a}$ if and only if all its first-order partial derivatives exist at $\mathbf{a}$. The following example shows that this guess is wrong even when the range of $\mathbf{f}$ is one dimensional.

Proof. If $(x, y) \neq (0, 0)$, then we can use the one-dimensional Product Rule to verify that both f_x and f_y exist and are continuous, for example,

$$f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \cos \frac{1}{\sqrt{x^2 + y^2}} + 2x \sin \frac{1}{\sqrt{x^2 + y^2}}.$$

Thus f is differentiable on $\mathbf{R}^2 \setminus \{(0, 0)\}$. Since $f_x(x, 0)$ has no limit as $x \to 0$, the partial derivative f_x is not continuous at $(0, 0)$. A similar statement holds for f_y. Thus to check differentiability at $(0, 0)$ we must return to the definition.

First, we compute the partial derivatives at $(0, 0)$. By definition,

$$f_x(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \to 0} t \sin \frac{1}{|t|} = 0,$$

and similarly, $f_y(0, 0) = 0$. Thus, both first partials exist at $(0, 0)$ and $\nabla f(0, 0) = (0, 0)$.

To prove that f is differentiable at $(0, 0)$, we must verify (4) for $\mathbf{a} = (0, 0)$. But it is clear that

$$\frac{f(h, k) - f(0, 0) - \nabla f(0, 0) \cdot (h, k)}{\|(h, k)\|} = \sqrt{h^2 + k^2} \sin \frac{1}{\sqrt{h^2 + k^2}} \to 0$$

as $(h, k) \to (0, 0)$. Thus f is differentiable at $(0, 0)$. ∎

EXERCISES

11.2.1. Suppose, for $j = 1, 2, \ldots, n$, that f_j are real functions continuously differentiable on the interval $(-1, 1)$. Prove that

$$g(\mathbf{x}) := f_1(x_1) \cdots f_n(x_n)$$

is differentiable on the cube $(-1, 1) \times (-1, 1) \times \cdots \times (-1, 1)$.

11.2.2. Suppose that $\mathbf{f}, \mathbf{g} : \mathbf{R} \to \mathbf{R}^m$ are differentiable at a and there is a $\delta > 0$ such that $\mathbf{g}(x) \neq \mathbf{0}$ for all $0 < |x - a| < \delta$. If $\mathbf{f}(a) = \mathbf{g}(a) = \mathbf{0}$ and $D\mathbf{g}(a) \neq \mathbf{0}$, prove that

$$\lim_{x \to a} \frac{\|\mathbf{f}(x)\|}{\|\mathbf{g}(x)\|} = \frac{\|D\mathbf{f}(a)\|}{\|D\mathbf{g}(a)\|}.$$

11.2.3. Prove that $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0, 0)$.

11.2.4. Prove that

$$f(x, y) = \begin{cases} \dfrac{x^2 + y^2}{\sin \sqrt{x^2 + y^2}} & 0 < \|(x, y)\| < \pi \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not differentiable at $(0, 0)$.

11.2.5. Prove that

$$f(x,y) = \begin{cases} \dfrac{x^4 + y^4}{(x^2+y^2)^\alpha} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable on $\mathbf{R}^2$ for all $\alpha < 3/2$.

11.2.6. Prove that if $\alpha > 1/2$, then

$$f(x,y) = \begin{cases} |xy|^\alpha \log(x^2+y^2) & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is differentiable at $(0,0)$.

11.2.7. Prove that

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbf{R}^2$ and has first-order partial derivatives everywhere on $\mathbf{R}^2$, but f is not differentiable at $(0,0)$.

$\boxed{\textbf{11.2.8}}$. **This exercise is used several times in this chapter and the next.** Suppose that $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$. Prove that $\mathbf{T}$ is differentiable everywhere on $\mathbf{R}^n$ with

$$D\mathbf{T}(\mathbf{a}) = \mathbf{T} \quad \text{for} \quad \mathbf{a} \in \mathbf{R}^n.$$

11.2.9. Let $r > 0$, $f : B_r(\mathbf{0}) \to \mathbf{R}$, and suppose that there exists an $\alpha > 1$ such that $|f(\mathbf{x})| \leq \|\mathbf{x}\|^\alpha$ for all $\mathbf{x} \in B_r(\mathbf{0})$. Prove that f is differentiable at $\mathbf{0}$. What happens to this result when $\alpha = 1$?

11.2.10. Let V be open in $\mathbf{R}^n$, $\mathbf{a} \in V$, and $\mathbf{f} : V \to \mathbf{R}^m$.

***11.19 Definition.**

If $\mathbf{u}$ is a *unit* vector in $\mathbf{R}^n$ (i.e., $\|\mathbf{u}\| = 1$), then the *directional derivative* of $\mathbf{f}$ at $\mathbf{a}$ in the direction $\mathbf{u}$ is defined by

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{a}) := \lim_{t\to 0} \frac{\mathbf{f}(\mathbf{a}+t\mathbf{u}) - \mathbf{f}(\mathbf{a})}{t}$$

when this limit exists.

a) Prove that $D_{\mathbf{u}}\mathbf{f}(\mathbf{a})$ exists for $\mathbf{u} = \mathbf{e}_k$ if and only if $\mathbf{f}_{x_k}(\mathbf{a})$ exists, in which case

$$D_{\mathbf{e}_k}\mathbf{f}(\mathbf{a}) = \frac{\partial \mathbf{f}}{\partial x_k}(\mathbf{a}).$$

b) Show that if $\mathbf{f}$ has directional derivatives at $\mathbf{a}$ in all directions $\mathbf{u}$, then the first-order partial derivatives of $\mathbf{f}$ exist at $\mathbf{a}$. Use Example 11.11 to show that the converse of this statement is false.

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c) Prove that the directional derivatives of

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

exist at $(0,0)$ in all directions $\mathbf{u}$, but f is neither continuous nor differentiable at $(0,0)$.

11.2.11. Let $r > 0$, $(a,b) \in \mathbf{R}^2$, $f : B_r(a,b) \to \mathbf{R}$, and suppose that the first-order partial derivatives f_x and f_y exist in $B_r(a,b)$ and are differentiable at (a,b).

a) Set $\Delta(h) = f(a+h, b+h) - f(a+h, b) - f(a, b+h) + f(a,b)$ and prove for h sufficiently small that

$$\begin{aligned}\frac{\Delta(h)}{h} &= f_y(a+h, b+th) - f_y(a,b) - \nabla f_y(a,b) \cdot (h, th) \\ &\quad - \big(f_y(a, b+th) - f_y(a,b) - \nabla f_y(a,b) \cdot (0, th)\big) + h f_{yx}(a,b)\end{aligned}$$

for some $t \in (0,1)$.

b) Prove that

$$\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a,b).$$

c) Prove that

$$\frac{\partial^2 f}{\partial x\, \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b).$$

11.3 DERIVATIVES, DIFFERENTIALS, AND TANGENT PLANES

In this section we begin to explore the analogy between $D\mathbf{f}$ and f'. First we examine how the total derivative interacts with the algebra of functions.

11.20 Theorem. *Let $\alpha \in \mathbf{R}$, $\mathbf{a} \in \mathbf{R}^n$, and suppose that $\mathbf{f}$ and $\mathbf{g}$ are vector functions. If $\mathbf{f}$ and $\mathbf{g}$ are differentiable at $\mathbf{a}$, then $\mathbf{f}+\mathbf{g}$, $\alpha\mathbf{f}$, and $\mathbf{f}\cdot\mathbf{g}$ are all differentiable at $\mathbf{a}$. In fact,*

$$D(\mathbf{f}+\mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) + D\mathbf{g}(\mathbf{a}), \tag{7}$$

$$D(\alpha\mathbf{f})(\mathbf{a}) = \alpha D\mathbf{f}(\mathbf{a}), \tag{8}$$

and

$$D(\mathbf{f}\cdot\mathbf{g})(\mathbf{a}) = \mathbf{g}(\mathbf{a}) D\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{a}) D\mathbf{g}(\mathbf{a}). \tag{9}$$

[The sums which appear on the right side of (7) and (9) represent matrix addition, and the products which appear on the right side of (9) represent matrix multiplication.]

Since $Q \to 0$ as $\mathbf{h} \to \mathbf{0}$, we conclude by the Squeeze Theorem that $\varepsilon(\mathbf{h})/\|\mathbf{h}\| \to 0$ as $\mathbf{h} \to \mathbf{0}$. In particular, f is differentiable at (a, b) and $(f_x(a, b), f_y(a, b), -1)$ is normal to its tangent plane there. ■

EXERCISES

11.3.1. For each of the following, prove that $\mathbf{f}$ and $\mathbf{g}$ are differentiable on their domains, and find formulas for $D(\mathbf{f} + \mathbf{g})(\mathbf{x})$ and $D(\mathbf{f} \cdot \mathbf{g})(\mathbf{x})$.

a) $$\mathbf{f}(x, y) = x - y, \quad \mathbf{g}(x, y) = x^2 + y^2$$

b) $$\mathbf{f}(x, y) = xy, \quad \mathbf{g}(x, y) = x \sin x - \cos y$$

c) $$\mathbf{f}(x, y) = (\cos(xy), x \log y), \quad \mathbf{g}(x, y) = (y, x)$$

d) $$\mathbf{f}(x, y, z) = (y, x - z), \quad \mathbf{g}(x, y, z) = (xyz, y^2)$$

11.3.2. For each of the following functions, find an equation of the tangent plane to $z = f(x, y)$ at $\mathbf{c}$.

a) $f(x, y) = x^2 + y^2$, $\mathbf{c} = (1, -1, 2)$
b) $f(x, y) = x^3 y - xy^3$, $\mathbf{c} = (1, 1, 0)$
c) $f(x, y, z) = xy + \sin z$, $\mathbf{c} = (1, 0, \pi/2, 1)$

11.3.3. Find all points on the paraboloid $z = x^2 + y^2$ (see Appendix D) where the tangent plane is parallel to the plane $x + y + z = 1$. Find equations of the corresponding tangent planes. Sketch the graphs of these functions to see that your answer agrees with your intuition.

11.3.4. Let $\mathcal{K}$ be the cone, given by $z = \sqrt{x^2 + y^2}$.

a) Find an equation of each plane tangent to $\mathcal{K}$ which is perpendicular to the plane $x + z = 5$.
b) Find an equation of each plane tangent to $\mathcal{K}$ which is parallel to the plane $x - y + z = 1$.

11.3.5. Prove (7) and (8) in Theorem 11.20.

11.3.6. [Quotient Rule]. Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ is differentiable at $\mathbf{a}$ and that $f(\mathbf{a}) \neq 0$.

a) Show that for $\|\mathbf{h}\|$ sufficiently small, $f(\mathbf{a} + \mathbf{h}) \neq 0$.
b) Prove that $Df(\mathbf{a})(\mathbf{h})/\|\mathbf{h}\|$ is bounded for all $\mathbf{h} \in \mathbf{R}^n \setminus \{\mathbf{0}\}$.
c) If $T := -Df(\mathbf{a})/f^2(\mathbf{a})$, show that

$$\frac{1}{f(\mathbf{a}+\mathbf{h})} - \frac{1}{f(\mathbf{a})} - T(\mathbf{h}) = \frac{f(\mathbf{a}) - f(\mathbf{a}+\mathbf{h}) + Df(\mathbf{a})(\mathbf{h})}{f(\mathbf{a})f(\mathbf{a}+\mathbf{h})} + \frac{(f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}))Df(\mathbf{a})(\mathbf{h})}{f^2(\mathbf{a})f(\mathbf{a}+\mathbf{h})}$$

for $\|\mathbf{h}\|$ sufficiently small.

d) Prove that $1/f(\mathbf{x})$ is differentiable at $\mathbf{x} = \mathbf{a}$ and

$$D\left(\frac{1}{f}\right)(\mathbf{a}) = -\frac{Df(\mathbf{a})}{f^2(\mathbf{a})}.$$

e) Prove that if f and g are real-valued vector functions which are differentiable at some $\mathbf{a}$, and if $g(\mathbf{a}) \neq 0$, then

$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g^2(\mathbf{a})}.$$

11.3.7. [CROSS-PRODUCT RULE]. Suppose that V is open in $\mathbf{R}^n$, that $\mathbf{f}, \mathbf{g} : V \to \mathbf{R}^3$, and that $\mathbf{a} \in V$. If $\mathbf{f}$ and $\mathbf{g}$ are differentiable at $\mathbf{a}$, prove that $\mathbf{f} \times \mathbf{g}$ is differentiable at $\mathbf{a}$ and

$$D(\mathbf{f} \times \mathbf{g})(\mathbf{a})(\mathbf{y}) = \mathbf{f}(\mathbf{a}) \times (D\mathbf{g}(\mathbf{a})(\mathbf{y})) - \mathbf{g}(\mathbf{a}) \times (D\mathbf{f}(\mathbf{a})(\mathbf{y}))$$

for all $\mathbf{y} \in \mathbf{R}^n$.

***11.3.8.** Compute the differential of each of the following functions.

a) $z = x^2 + y^2$ b) $z = \sin(xy)$ c) $z = \dfrac{xy}{1 + x^2 + y^2}$

***11.3.9.** Let $w = x^2y + z$. Use differentials to approximate Δw as (x, y, z) moves from (1,2,1) to (1.01, 1.98, 1.03). Compare your approximation with the actual value of Δw.

***11.3.10.** The time T it takes for a pendulum to complete one full swing is given by

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the acceleration due to gravity and L is the length of the pendulum. If g can be measured with a maximum error of 1%, how accurately must L be measured (in terms of percentage error) so that the calculated value of T has a maximum error of 2%?

***11.3.11.** Suppose that

$$\frac{1}{w} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

where each variable x, y, z can be measured with a maximum error of $p\%$. Prove that the calculated value of w also has a maximum error of $p\%$.

11.4 THE CHAIN RULE

Here is the Chain Rule for vector functions.

The Chain Rule can be used to compute individual partial derivatives without writing out the entire matrices $D\mathbf{f}$ and $D\mathbf{g}$. For example, suppose that $f(u_1, \dots, u_m)$ is differentiable from $\mathbf{R}^m$ to $\mathbf{R}$, that $\mathbf{g}(x_1, \dots, x_n)$ is differentiable from $\mathbf{R}^n$ to $\mathbf{R}^m$, and that $z = f(\mathbf{g}(x_1, \dots, x_n))$. Since $Df = \nabla f$ and the jth column of $D\mathbf{g}$ consists of first partial derivatives, with respect to x_j, of the components $u_k := g_k(x_1, \dots, x_n)$, it follows from the Chain Rule and the definition of matrix multiplication that

$$\frac{\partial z}{\partial x_j} = \frac{\partial f}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \dots + \frac{\partial f}{\partial u_m}\frac{\partial u_m}{\partial x_j}$$

for $j = 1, 2, \dots, n$. Here are two concrete examples which illustrate this principle.

11.29 *EXAMPLES.*

i) If $F, G, H : \mathbf{R}^2 \to \mathbf{R}$ are differentiable and $z = F(x, y)$, where $x = G(r, \theta)$, and $y = H(r, \theta)$, then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} \quad \text{and} \quad \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}.$$

ii) If $f : \mathbf{R}^3 \to \mathbf{R}$ and $\phi, \psi, \sigma : \mathbf{R} \to \mathbf{R}$ are differentiable and $w = f(x, y, z)$, where $x = \phi(t)$, $y = \psi(t)$, and $z = \sigma(t)$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

EXERCISES

11.4.1. Let $F : \mathbf{R}^3 \to \mathbf{R}$ and $f, g, h : \mathbf{R}^2 \to \mathbf{R}$ be C^2 functions. If $w = F(x, y, z)$, where $x = f(p, q)$, $y = g(p, q)$, and $z = h(p, q)$, find formulas for w_p, w_q, and w_{pp}.

11.4.2. Let $r > 0$, let $\mathbf{a} \in \mathbf{R}^n$, and suppose that $\mathbf{g} : B_r(\mathbf{a}) \to \mathbf{R}^m$ is differentiable at $\mathbf{a}$.

a) If $f : B_r(g(\mathbf{a})) \to \mathbf{R}$ is differentiable at $\mathbf{g}(\mathbf{a})$, prove that the partial derivatives of $h = f \circ \mathbf{g}$ are given by

$$\frac{\partial h}{\partial x_j}(\mathbf{a}) = \nabla f(\mathbf{g}(\mathbf{a})) \cdot \frac{\partial \mathbf{g}}{\partial x_j}(\mathbf{a})$$

for $j = 1, 2, \dots, n$.

b) If $n = m$ and $\mathbf{f} : B_r(g(\mathbf{a})) \to \mathbf{R}^n$ is differentiable at $\mathbf{g}(\mathbf{a})$, prove that

$$\det(D(\mathbf{f} \circ \mathbf{g})(\mathbf{a})) = \det(D\mathbf{f}(\mathbf{g}(\mathbf{a}))) \det(D\mathbf{g}(\mathbf{a})).$$

11.4.3. Suppose that $k \in \mathbf{N}$ and that $f : \mathbf{R}^n \to \mathbf{R}$ is homogeneous of order k; that is, that $f(\rho\mathbf{x}) = \rho^k f(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{R}^n$ and all $\rho \in \mathbf{R}$. If f is differentiable on $\mathbf{R}^n$, prove that

$$x_1 \frac{\partial f}{\partial x_1}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = kf(\mathbf{x})$$

for all $\mathbf{x} = (x_1, \dots, x_n) \in \mathbf{R}^n$.

11.4.4. Let $f, g : \mathbf{R} \to \mathbf{R}$ be twice differentiable. Prove that $u(x, y) := f(xy)$ satisfies

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0,$$

and $v(x, y) := f(x - y) + g(x + y)$ satisfies the *wave equation*; that is,

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

11.4.5. Let $f, g : \mathbf{R}^2 \to \mathbf{R}$ be differentiable and satisfy the *Cauchy–Riemann equations*; that is, that

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

hold on $\mathbf{R}^2$. If $u(r, \theta) = f(r\cos\theta, r\sin\theta)$ and $v(r, \theta) = g(r\cos\theta, r\sin\theta)$, prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad r \neq 0.$$

11.4.6. Let $f : \mathbf{R}^2 \to \mathbf{R}$ be $\mathcal{C}^2$ on $\mathbf{R}^2$ and set $u(r, \theta) = f(r\cos\theta, r\sin\theta)$. If f satisfies the *Laplace equation*; that is, if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

prove for each $r \neq 0$ that

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = 0.$$

11.4.7. Let

$$u(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}, \qquad t > 0,\ x \in \mathbf{R}.$$

a) Prove that u satisfies the *heat equation* (i.e., $u_{xx} - u_t = 0$ for all $t > 0$ and $x \in \mathbf{R}$).

b) If $a > 0$, prove that $u(x,t) \to 0$, as $t \to 0+$, uniformly for $x \in [a, \infty)$.

11.4.8. Let $u : \mathbf{R} \to [0, \infty)$ be differentiable. Prove that for each $(x, y, z) \neq (0, 0, 0)$,

$$F(x, y, z) := u\left(\sqrt{x^2 + y^2 + z^2}\right)$$

satisfies

$$\left(\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right)^{1/2} = \left|u'\left(\sqrt{x^2 + y^2 + z^2}\right)\right|.$$

11.4.9. Suppose that $z = F(x, y)$ is differentiable at (a, b), that $F_y(a, b) \neq 0$, and that I is an open interval containing a. Prove that if $f : I \to \mathbf{R}$ is differentiable at a, $f(a) = b$, and $F(x, f(x)) = 0$ for all $x \in I$, then

$$\frac{df}{dx}(a) = \frac{-\dfrac{\partial F}{\partial x}(a, b)}{\dfrac{\partial F}{\partial y}(a, b)}.$$

11.4.10. Suppose that I is a nonempty, open interval and that $\mathbf{f} : I \to \mathbf{R}^m$ is differentiable on I. If $\mathbf{f}(I) \subseteq \partial B_r(\mathbf{0})$ for some fixed $r > 0$, prove that $\mathbf{f}(t)$ is orthogonal to $\mathbf{f}'(t)$ for all $t \in I$.

11.4.11. Let V be open in $\mathbf{R}^n$, $\mathbf{a} \in V$, $f : V \to \mathbf{R}$, and suppose that f is differentiable at $\mathbf{a}$.

a) Prove that the directional derivative $D_{\mathbf{u}}f(\mathbf{a})$ exists (see Exercise 11.2.10) for each $\mathbf{u} \in \mathbf{R}^n$ such that $\|\mathbf{u}\| = 1$ and $D_{\mathbf{u}}f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$.

b) If $\nabla f(\mathbf{a}) \neq \mathbf{0}$ and θ represents the angle between $\mathbf{u}$ and $\nabla f(\mathbf{a})$, prove that $D_{\mathbf{u}}f(\mathbf{a}) = \|\nabla f(\mathbf{a})\| \cos\theta$.

c) Show that as $\mathbf{u}$ ranges over all unit vectors in $\mathbf{R}^n$, the maximum of $D_{\mathbf{u}}f(\mathbf{a})$ is $\|\nabla f(\mathbf{a})\|$, and it occurs when $\mathbf{u}$ is parallel to $\nabla f(\mathbf{a})$.

11.5 THE MEAN VALUE THEOREM AND TAYLOR'S FORMULA

Using $D\mathbf{f}$ as a replacement for f', we guess that the multidimensional analogue of the Mean Value Theorem is $\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{a}) = D\mathbf{f}(\mathbf{c})(\mathbf{x} - \mathbf{a})$ for some $\mathbf{c}$ "between" $\mathbf{x}$ and $\mathbf{a}$; that is, some $\mathbf{c} \in L(\mathbf{x}; \mathbf{a})$, the line segment from $\mathbf{a}$ to $\mathbf{x}$. The following result shows that this guess is correct when $\mathbf{f}$ is real valued (see also Exercises 11.5.6 and 11.5.9).

11.30 Theorem. [MEAN VALUE THEOREM FOR REAL VALUED FUNCTIONS].
Let V be open in $\mathbf{R}^n$ and suppose that $f : V \to \mathbf{R}$ is differentiable on V. If $\mathbf{x}, \mathbf{a} \in V$ and $L(\mathbf{x}; \mathbf{a}) \subset V$, then there is a $\mathbf{c} \in L(\mathbf{x}; \mathbf{a})$ such that

$$f(\mathbf{x}) - f(\mathbf{a}) = \nabla f(\mathbf{c}) \cdot (\mathbf{x} - \mathbf{a}). \tag{24}$$

Solution. It is easy to verify that f_x, f_y, f_{xx}, f_{xy}, and f_{yy} are all zero at $(0,0)$, so $D^{(1)}f((0,0);(x,y)) = 0$ and $D^{(2)}f((0,0);(x,y)) = 0$. Since $f_{xxx}(x,y) = y^3\sin(xy)$, $f_{xxy} = -2y\cos(xy) + xy^2\sin(xy)$, $f_{xyy} = -2x\cos(xy) + x^2y\sin(xy)$, and $f_{yyy} = x^3\cos(xy)$, Theorem 11.2 implies $D^{(3)}f((c,d);(x,y)) = (x^3 + y^3)\sin(cd) + 3(x^2y + xy^2)\sin(cd) - 6(x+y)\cos(cd)$. Thus by Taylor's Formula, for all $(x,y) \in \mathbf{R}^2$ there is a point (c,d) on the line segment between $(0,0)$ and (x,y) such that

$$\cos(xy) = 1 + \left(\frac{x^3+y^3}{6}\right)\sin(cd) + \left(\frac{x^2y+xy^2}{2}\right)\sin(cd) - (x+y)\cos(cd).$$

EXERCISES

11.5.1. a) Write out an expression in powers of $(x+1)$ and $(y-1)$ for $f(x,y) = x^2 + xy + y^2$.

b) Write Taylor's Formula for $f(x,y) = \sqrt{x} + \sqrt{y}$, $\mathbf{a} = (1,4)$, and $p = 3$.

c) Write Taylor's Formula for $f(x,y) = e^{xy}$, $\mathbf{a} = (0,0)$, and $p = 4$.

11.5.2. Suppose that $f : \mathbf{R}^2 \to \mathbf{R}$ is $\mathcal{C}^p$ on $B_r(x_0, y_0)$ for some $r > 0$. Prove that, given $(x,y) \in B_r(x_0,y_0)$, there is a point (c,d) on the line segment between (x_0,y_0) and (x,y) such that

$$f(x,y) = f(x_0,y_0) + \sum_{k=1}^{p-1}\frac{1}{k!}\left(\sum_{j=0}^{k}\binom{k}{j}(x-x_0)^j(y-y_0)^{k-j}\frac{\partial^k f}{\partial x^j\,\partial y^{k-j}}(x_0,y_0)\right)$$
$$+\frac{1}{p!}\sum_{j=0}^{p}\binom{p}{j}(x-x_0)^j(y-y_0)^{p-j}\frac{\partial^p f}{\partial x^j\,\partial y^{p-j}}(c,d).$$

11.5.3. Suppose that $f : \mathbf{R}^n \to \mathbf{R}$ and $\mathbf{g} : \mathbf{R}^n \to \mathbf{R}^n$ are differentiable on $\mathbf{R}^n$ and that there exist $r > 0$ and $\mathbf{a} \in \mathbf{R}^n$ such that $D\mathbf{g}(\mathbf{x})$ is the identity matrix, I, for all $\mathbf{x} \in B_r(\mathbf{a})$. Prove that there is a function $\mathbf{h} : B_r(\mathbf{a}) \setminus \{\mathbf{a}\} \to B_r(\mathbf{x})$ such that

$$\frac{|f(\mathbf{g}(\mathbf{x})) - f(\mathbf{g}(\mathbf{a}))|}{\|\mathbf{x}-\mathbf{a}\|} \le \|Df((\mathbf{g}\circ\mathbf{h})(\mathbf{x}))\|$$

for all $\mathbf{x} \in B_r(\mathbf{a}) \setminus \{\mathbf{a}\}$.

11.5.4. Suppose that V is convex and open in $\mathbf{R}^n$ and that $\mathbf{f} : V \to \mathbf{R}^n$ is differentiable on V. If there exists an $\mathbf{a} \in V$ such that $D\mathbf{f}(\mathbf{x}) = D\mathbf{f}(\mathbf{a})$ for all $\mathbf{x} \in V$, prove that there exist a linear function $\mathbf{S} \in \mathcal{L}(\mathbf{R}^n;\mathbf{R}^n)$ and a vector $\mathbf{c} \in \mathbf{R}^n$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{S}(\mathbf{x}) + \mathbf{c}$ for all $\mathbf{x} \in V$.

11.5.5. [Integral Form of Taylor's Formula]. Let $p \in \mathbf{N}$, V be an open set in $\mathbf{R}^n$, $\mathbf{x}, \mathbf{a} \in V$, and $f : V \to \mathbf{R}$ be $\mathcal{C}^p$ on V. If $L(\mathbf{x};\mathbf{a}) \subset V$ and $\mathbf{h} = \mathbf{x} - \mathbf{a}$, prove that

$$f(\mathbf{x})-f(\mathbf{a})=\sum_{k=1}^{p-1}\frac{1}{k!}D^{(k)}f(\mathbf{a};\mathbf{h})+\frac{1}{(p-1)!}\int_0^1(1-t)^{p-1}D^{(p)}f(\mathbf{a}+t\mathbf{h};\mathbf{h})\,dt.$$

11.5.6. Let $r>0$, $a,b\in\mathbf{R}$, $f:B_r(a,b)\to\mathbf{R}$ be differentiable, and $(x,y)\in B_r(a,b)$.

a) Let $g(t)=f(tx+(1-t)a,y)+f(a,ty+(1-t)b)$ and compute the derivative of g.

b) Prove that there are numbers c between a and x, and d between b and y such that

$$f(x,y)-f(a,b)=(x-a)f_x(c,y)+(y-b)f_y(a,d).$$

(This is Exercise 12.20 in Apostol [1].)

11.5.7. Suppose that $0<r<1$ and that $f:B_1(\mathbf{0})\to\mathbf{R}$ is continuously differentiable. If there is an $\alpha>0$ such that $|f(\mathbf{x})|\le\|\mathbf{x}\|^\alpha$ for all $\mathbf{x}\in B_r(\mathbf{0})$, prove that there is an $M>0$ such that $|f(\mathbf{x})|\le M\|\mathbf{x}\|$ for $\mathbf{x}\in B_r(\mathbf{0})$.

11.5.8. Suppose that V is open in $\mathbf{R}^n$, that $f:V\to\mathbf{R}$ is $\mathcal{C}^2$ on V, and that $f_{x_j}(\mathbf{a})=0$ for some $\mathbf{a}\in H$ and all $j=1,\dots,n$. Prove that if H is a compact convex subset of V, then there is a constant M such that

$$|f(\mathbf{x})-f(\mathbf{a})|\le M\|\mathbf{x}-\mathbf{a}\|^2$$

for all $\mathbf{x}\in H$.

11.5.9. Let $f:\mathbf{R}^n\to\mathbf{R}$. Suppose that for each unit vector $\mathbf{u}\in\mathbf{R}^n$, the directional derivative $D_{\mathbf{u}}f(\mathbf{a}+t\mathbf{u})$ exists for $t\in[0,1]$ (see Definition 11.19). Prove that

$$f(\mathbf{a}+\mathbf{u})-f(\mathbf{a})=D_{\mathbf{u}}f(\mathbf{a}+t\mathbf{u})$$

for some $t\in(0,1)$.

11.5.10. Suppose that V is open in $\mathbf{R}^2$, that $(a,b)\in V$, and that $f:V\to\mathbf{R}$ is $\mathcal{C}^3$ on V. Prove that

$$\lim_{r\to0}\frac{4}{\pi r^2}\int_0^{2\pi}f(a+r\cos\theta,b+r\sin\theta)\cos(2\theta)\,d\theta=f_{xx}(a,b)-f_{yy}(a,b).$$

11.5.11. Suppose that V is open in $\mathbf{R}^2$, that $H=[a,b]\times[0,c]\subset V$, that $u:V\to\mathbf{R}$ is $\mathcal{C}^2$ on V, and that $u(x_0,t_0)\ge0$ for all $(x_0,t_0)\in\partial H$.

a) Show that, given $\varepsilon>0$, there is a compact set $K\subset H^o$ such that $u(x,t)\ge-\varepsilon$ for all $(x,t)\in H\setminus K$.

b) Suppose that $u(x_1,t_1)=-\ell<0$ for some $(x_1,t_1)\in H^o$, and choose $r>0$ so small that $2rt_1<\ell$. Apply part a) to $\varepsilon:=\ell/2-rt_1$ to choose the compact set K, and prove that the minimum of

$$w(x,t):=u(x,t)+r(t-t_1)$$

on H occurs at some $(x_2,t_2)\in K$.

c) Prove that if u satisfies the *heat equation* (i.e., $u_{xx} - u_t = 0$ on V), and if $u(x_0, t_0) \geq 0$ for all $(x_0, t_0) \in \partial H$, then $u(x, t) \geq 0$ for all $(x, t) \in H$.

11.5.12. a) Prove that every convex set in $\mathbf{R}^n$ is connected.
b) Show that the converse of part a) is false.
*c) Suppose that $f : \mathbf{R} \to \mathbf{R}$. Prove that f is convex (as a function) if and only if $E := \{(x, y) : y \geq f(x)\}$ is convex (as a set in $\mathbf{R}^2$).

11.6 THE INVERSE FUNCTION THEOREM

By the one-dimensional Inverse Function Theorem (Theorem 4.33), if $g : \mathbf{R} \to \mathbf{R}$ is 1–1 and differentiable with $g'(x_0) \neq 0$, then g^{-1} is differentiable at $y_0 = g(x_0)$ and

$$(g^{-1})'(y_0) = \frac{1}{g'(x_0)}.$$

In this section we obtain a multivariable analogue of this result (i.e., an Inverse Function Theorem for vector functions $\mathbf{f}$ from n variables to n variables). What shall we use for hypotheses? We needed g to be 1–1 so that the inverse function g^{-1} existed. For the same reason, we shall assume that $\mathbf{f}$ is 1–1. We needed $g'(x_0)$ to be nonzero so that we could divide by it. In the multidimensional case, $D\mathbf{f}(\mathbf{a})$ is a matrix; hence "divisibility" corresponds to invertibility. Since an $n \times n$ matrix is invertible if and only if it has a nonzero determinant (see Appendix C), we shall assume that the *Jacobian* of $\mathbf{f}$

$$\Delta_{\mathbf{f}}(\mathbf{a}) := \det(D\mathbf{f}(\mathbf{a})) \neq 0.$$

The word *Jacobian* is used because it was Jacobi who first recognized the importance of $\Delta_{\mathbf{f}}$ and its connection with volume (see Exercise 12.4.6).

The proof of the Inverse Function Theorem on $\mathbf{R}^n$ is not simple and lies somewhat deeper than the preceding results of this chapter. Before presenting it, we first prove two preliminary results which explore the consequences of the hypothesis $\Delta_{\mathbf{f}} \neq 0$.

11.39 ***Lemma.***
Let V be open and nonempty in $\mathbf{R}^n$ and let $\mathbf{f} : V \to \mathbf{R}^n$ be continuous. If $\mathbf{f}$ is 1–1 and has first-order partial derivatives on V, and if $\Delta_{\mathbf{f}} \neq 0$ on V, then $\mathbf{f}^{-1}$ is continuous on $\mathbf{f}(V)$.

STRATEGY: To prove that $\mathbf{f}^{-1}$ is continuous on V, it suffices to prove (apply Exercise 9.4.3 or Theorem 10.58 to $\mathbf{f}^{-1}$) that $\mathbf{f}(W) = (\mathbf{f}^{-1})^{-1}(W)$ is open for all open $W \subseteq V$. Thus, given $\mathbf{b} \in \mathbf{f}(W)$, say $\mathbf{b} = \mathbf{f}(\mathbf{a})$ for some $\mathbf{a} \in W$, we must find a $\rho > 0$ such that $B_\rho(\mathbf{b}) \subseteq \mathbf{f}(W)$. We will actually show more: that if $\overline{B_r(\mathbf{a})} \subset W$ for some $r > 0$, then there is a $\rho > 0$ such that $B_\rho(\mathbf{b}) \subseteq \mathbf{f}(B_r(\mathbf{a}))$; that is, $\mathbf{y} \in B_\rho(\mathbf{f}(\mathbf{a}))$ implies that $\mathbf{y} = \mathbf{f}(\mathbf{c})$ for some $\mathbf{c} \in B_r(\mathbf{a})$.

Where should we look to find such a $\mathbf{c}$? To show that $\mathbf{f}(\mathbf{c}) - \mathbf{y} = \mathbf{0}$, we might first try finding a point $\mathbf{c} \in B_r(\mathbf{a})$ that minimizes $\|\mathbf{f}(\mathbf{x}) - \mathbf{y}\|$ as $\mathbf{x}$ ranges over $\overline{B_r(\mathbf{a})}$

Then $\mathbf{F}(4, 3, 2, 1, -1, -2) = (0, 0)$, and

$$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \det\begin{bmatrix} 2u & 2v \\ 2u/x^2 & 2v/y^2 \end{bmatrix} = 4uv\left(\frac{1}{y^2} - \frac{1}{x^2}\right).$$

This determinant is nonzero when $u = 4$, $v = 3$, $x = 2$, and $y = 1$. Therefore, such functions u, v exist by the Implicit Function Theorem. ∎

EXERCISES

11.6.1. For each of the following functions, prove that $\mathbf{f}^{-1}$ exists and is differentiable in some nonempty, open set containing (a, b), and compute $D(\mathbf{f}^{-1})(a, b)$

a) $\mathbf{f}(u, v) = (3u - v, 2u + 5v)$ at any $(a, b) \in \mathbf{R}^2$
b) $\mathbf{f}(u, v) = (u + v, \sin u + \cos v)$ at $(a, b) = (0, 1)$
c) $\mathbf{f}(u, v) = (uv, u^2 + v^2)$ at $(a, b) = (2, 5)$
d) $\mathbf{f}(u, v) = (u^3 - v^2, \sin u - \log v)$ at $(a, b) = (-1, 0)$

11.6.2. For each of the following functions, find out whether the given expression can be solved for z in a nonempty, open set V containing $(0, 0, 0)$. Is the solution differentiable near $(0, 0)$?

a) $xyz + \sin(x + y + z) = 0$
b) $x^2 + y^2 + z^2 + \sqrt{\sin(x^2 + y^2) + 3z + 4} = 2$
c) $xyz(2\cos y - \cos z) + (z\cos x - x\cos y) = 0$
d) $x + y + z + g(x, y, z) = 0$, where g is any continuously differentiable function which satisfies $g(0, 0, 0) = 0$ and $g_z(0, 0, 0) > 0$

11.6.3. Prove that there exist functions $u(x, y)$, $v(x, y)$, and $w(x, y)$, and an $r > 0$ such that u,v,w are continuously differentiable and satisfy the equations

$$u^5 + xv^2 - y + w = 0$$
$$v^5 + yu^2 - x + w = 0$$
$$w^4 + y^5 - x^4 = 1$$

on $B_r(1, 1)$, and $u(1, 1) = 1$, $v(1, 1) = 1$, $w(1, 1) = -1$.

11.6.4. Find conditions on a point (x_0, y_0, u_0, v_0) such that there exist real-valued functions $u(x, y)$ and $v(x, y)$ which are continuously differentiable near (x_0, y_0) and satisfy the simultaneous equations

$$xu^2 + yv^2 + xy = 9$$
$$xv^2 + yu^2 - xy = 7.$$

Prove that the solutions satisfy $u^2 + v^2 = 16/(x + y)$.

11.6.5. Given nonzero numbers $x_0, y_0, u_0, v_0, s_0, t_0$ which satisfy the simultaneous equations

$$(*)\qquad \begin{aligned} u^2 + sx + ty &= 0 \\ v^2 + tx + sy &= 0 \\ 2s^2x + 2t^2y - 1 &= 0 \\ s^2x - t^2y &= 0, \end{aligned}$$

prove that there exist functions $u(x, y),\ v(x, y),\ s(x, y),\ t(x, y)$, and an open ball B containing (x_0, y_0), such that u,v,s,t are continuously differentiable and satisfy (*) on B, and such that $u(x_0, y_0) = u_0,\ v(x_0, y_0) = v_0,\ s(x_0, y_0) = s_0$, and $t(x_0, y_0) = t_0$.

11.6.6. Let $E = \{(x, y) : 0 < y < x\}$ and set $\mathbf{f}(x, y) = (x + y, xy)$ for $(x, y) \in E$.

a) Prove that $\mathbf{f}$ is 1–1 from E onto $\{(s, t) : s > 2\sqrt{t}, t > 0\}$ and find a formula for $\mathbf{f}^{-1}(s, t)$.
b) Use the Inverse Function Theorem to compute $D(\mathbf{f}^{-1})(\mathbf{f}(x, y))$ for $(x, y) \in E$.
c) Use the formula you obtained in part a) to compute $D(\mathbf{f}^{-1})(s, t)$ directly. Check to see that this answer agrees with the one you found in part b).

11.6.7. Suppose that V is open in $\mathbf{R}^n$, that $\mathbf{a} \in V$, and that $F : V \to \mathbf{R}$ is $\mathcal{C}^1$ on V. If $F(\mathbf{a}) = 0 \neq F_{x_j}(\mathbf{a})$ and $\mathbf{u}^{(j)} := (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n)$ for $j = 1, 2, \dots, n$, prove that there exist open sets W_j containing $(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n)$, an $r > 0$, and functions $g_j(\mathbf{u}^{(j)})$, $\mathcal{C}^1$ on W_j, such that $F(x_1, \dots, x_{j-1}, g_j(\mathbf{u}^{(j)}), x_{j+1}, \dots, x_n) = 0$ on W_j and

$$\frac{\partial g_1}{\partial x_n}\frac{\partial g_2}{\partial x_1}\frac{\partial g_3}{\partial x_2}\cdots\frac{\partial g_n}{\partial x_{n-1}} = (-1)^n$$

on $B_r(\mathbf{a})$.

11.6.8. Suppose that $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ has continuous first-order partial derivatives in some ball $B_r(x_0, y_0)$, $r > 0$. Prove that if $\Delta_{\mathbf{f}}(x_0, y_0) \neq 0$, then

$$\frac{\partial f_1^{-1}}{\partial x}(f(x_0, y_0)) = \frac{\partial f_2/\partial y(x_0, y_0)}{\Delta_{\mathbf{f}}(x_0, y_0)}, \quad \frac{\partial f_1^{-1}}{\partial y}(f(x_0, y_0)) = \frac{-\partial f_1/\partial y(x_0, y_0)}{\Delta_{\mathbf{f}}(x_0, y_0)},$$

and

$$\frac{\partial f_2^{-1}}{\partial x}(f(x_0, y_0)) = \frac{-\partial f_2/\partial x(x_0, y_0)}{\Delta_{\mathbf{f}}(x_0, y_0)}, \quad \frac{\partial f_2^{-1}}{\partial y}(f(x_0, y_0)) = \frac{\partial f_1/\partial x(x_0, y_0)}{\Delta_{\mathbf{f}}(x_0, y_0)}.$$

$\boxed{\textbf{11.6.9}}$. **This exercise is used in Section *11.7**. Let $F : \mathbf{R}^3 \to \mathbf{R}$ be continuously differentiable in some open set containing (a, b, c) with $F(a, b, c) = 0$ and $\nabla F(a, b, c) \neq 0$.

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a) Prove that the graph of the relation $F(x, y, z) = 0$; that is, that the set $\mathcal{G} := \{(x, y, z) : F(x, y, z) = 0\}$ has a tangent plane at (a, b, c).
b) Prove that a normal of the tangent plane to $\mathcal{G}$ at (a, b, c) is given by $\nabla F(a, b, c)$.

11.6.10. Suppose that $\mathbf{f} := (u, v) : \mathbf{R} \to \mathbf{R}^2$ is $\mathcal{C}^2$ and that $(x_0, y_0) = \mathbf{f}(t_0)$.

a) Prove that if $\mathbf{f}'(t_0) \neq \mathbf{0}$, then $u'(t_0)$ and $v'(t_0)$ cannot both be zero.
b) If $\mathbf{f}'(t_0) \neq 0$, show that either there is a $\mathcal{C}^1$ function g such that $g(x_0) = t_0$ and $u(g(x)) = x$ for x near x_0, or there is a $\mathcal{C}^1$ function h such that $h(y_0) = t_0$ and $v(h(y)) = y$ for y near y_0.

11.6.11. Let $\mathcal{H}$ be the hyperboloid of one sheet, given by $x^2 + y^2 - z^2 = 1$.

a) Use Exercise 11.6.9 to prove that at every point $(a, b, c) \in \mathcal{H}$, $\mathcal{H}$ has a tangent plane whose normal is given by $(-a, -b, c)$.
b) Find an equation of each plane tangent to $\mathcal{H}$ which is perpendicular to the xy-plane.
c) Find an equation of each plane tangent to $\mathcal{H}$ which is parallel to the plane $x + y - z = 1$.

*11.7 OPTIMIZATION

This section uses no material from any other enrichment section.

In this section we discuss how to find extreme values of differentiable functions of several variables.

11.50 Definition.

Let V be open in $\mathbf{R}^n$, let $\mathbf{a} \in V$, and suppose that $f : V \to \mathbf{R}$.

i) $f(\mathbf{a})$ is called a *local minimum* of f if and only if there is an $r > 0$ such that $f(\mathbf{a}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in B_r(\mathbf{a})$.
ii) $f(\mathbf{a})$ is called a *local maximum* of f if and only if there is an $r > 0$ such that $f(\mathbf{a}) \geq f(\mathbf{x})$ for all $\mathbf{x} \in B_r(\mathbf{a})$.
iii) $f(\mathbf{a})$ is called a *local extremum* of f if and only if $f(\mathbf{a})$ is a local maximum or a local minimum of f.

The following result shows that, as in the one-dimensional case, extrema of real-valued differentiable functions occur among points where the "derivative" is zero.

11.51 Remark. *If the first-order partial derivatives of f exist at* $\mathbf{a}$, *and $f(\mathbf{a})$ is a local extremum of f, then $\nabla f(\mathbf{a}) = \mathbf{0}$.*

Proof. The one-dimensional function $g(t) = f(a_1, \dots, a_{j-1}, t, a_{j+1}, \dots, a_n)$ has a local extremum at $t = a_j$ for each $j = 1, \dots, n$. Hence, by the one-dimensional theory,