for all \( y \in [c, d] \) which satisfy \( |y - y_0| < \delta \). Then
\[
|F(y) - F(y_0)| \leq \left| F(y) - \int_A^B f(x, y) \, dx \right| + \left| \int_A^B (f(x, y) - f(x, y_0)) \, dx \right| + \left| F(y_0) - \int_A^B f(x, y_0) \, dx \right|
\]
\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]
for all \( y \in [c, d] \) which satisfy \( |y - y_0| < \delta \).

The proof of Theorem 11.5 can be modified to prove the following result.

**11.9 Theorem.** Suppose that \( a < b \) are extended real numbers, that \( c < d \) are finite real numbers, that \( f : (a, b) \times [c, d] \to \mathbb{R} \) is continuous, and that the improper integral
\[
F(y) = \int_a^b f(x, y) \, dx
\]
exists for all \( y \in [c, d] \). If \( f_x(x, y) \) exists and is continuous on \((a, b) \times [c, d]\) and if
\[
\phi(y) = \int_a^b \frac{\partial f}{\partial y}(x, y) \, dx
\]
converges uniformly on \([c, d]\), then \( F \) is differentiable on \([c, d]\) and \( F'(y) = \phi(y) \); that is,
\[
\frac{d}{dy} \int_a^b f(x, y) \, dx = \int_a^b \frac{\partial f}{\partial y}(x, y) \, dx
\]
for all \( y \in [c, d] \).

For a result about interchanging two partial integrals, see Theorem 12.31 and Exercise 12.3.10.

**EXERCISES**

**11.1.** Compute all mixed second-order partial derivatives of each of the following functions and verify that the mixed partial derivatives are equal.

a) \( f(x, y) = xe^y \)  \quad b) \( f(x, y) = \cos(xy) \)  \quad c) \( f(x, y) = \frac{x + y}{x^2 + 1} \)

**11.2.** For each of the following functions, compute \( f_y \), and determine where it is continuous.
11.1.3. Suppose that \( r > 0 \), that \( \mathbf{a} \in \mathbb{R}^n \), and that \( f : B_r(\mathbf{a}) \rightarrow \mathbb{R}^n \). If all first-order partial derivatives of \( f \) exist on \( B_r(\mathbf{a}) \) and satisfy \( f_j'(x) = 0 \) for all \( x \in B_r(\mathbf{a}) \) and all \( j = 1, 2, \ldots, n \), prove that \( f \) has only one value on \( B_r(\mathbf{a}) \).

11.1.4. Suppose that \( H = [a, b] \times [c, d] \) is a rectangle, that \( f : H \rightarrow \mathbb{R} \) is continuous, and that \( g : [a, b] \rightarrow \mathbb{R} \) is integrable. Prove that

\[
F(y) = \int_a^b g(x) f(x, y) \, dx
\]

is uniformly continuous on \([c, d]\).

11.1.5. Evaluate each of the following expressions.

a) \[
\lim_{y \to 0} \int_0^1 e^{x^2y^2 + x} \, dx
\]

b) \[
\frac{d}{dy} \int_0^1 \sin(e^y - y^3 + \pi - e^x) \, dx \quad \text{at } y = 1
\]

c) \[
\frac{\partial}{\partial x} \int_1^3 \sqrt{x^3 + y^3 + z^3} - 2 \, dz \quad \text{at } (x, y) = (1, 1)
\]

11.1.6. Suppose that \( f \) is a continuous real function.

a) If \( \int_0^1 f(x) \, dx = 1 \), find the exact value of

\[
\lim_{y \to 0} \int_0^2 f((x - 1)e^{x^2y^2 + x}) \, dx.
\]

b) If \( f \) is \( C^1 \) on \( \mathbb{R} \) and \( \int_0^\pi f'(x) \sin x \, dx = e \), find the exact value of

\[
e + \lim_{y \to 0} \int_0^\pi f(x) \cos(y^5 + \sqrt{y} + x) \, dx.
\]

c) If \( \int_0^1 f(\sqrt{x})e^x \, dx = 6 \), find the exact value of

\[
\frac{d}{dx} \int_0^1 f(y)e^{xy + y^2} \, dy \quad \text{at } x = 0.
\]
*11.1.7. Evaluate each of the following expressions.

a) \[ \lim_{y \to 0^+} \int_0^1 \frac{x \cos \frac{y}{x}}{\sqrt{1 - x + y}} \, dx \]

b) \[ \frac{d}{dy} \int_\pi^\infty e^{-y^2} \sin \frac{x}{x} \, dx \quad \text{at} \ y = 1 \]

*11.1.8. a) Prove that

\[ \int_0^1 \frac{\cos(x^2 + y^2)}{\sqrt{x}} \, dx \]

converges uniformly on \((−\infty, \infty)\).

b) Prove that \(\int_0^\infty e^{-xy} \, dx\) converges uniformly on \([1, \infty)\).

c) Prove that \(\int_0^\infty ye^{-xy} \, dx\) exists for each \(y \in [0, \infty)\) and converges uniformly on any \([a, b] \subset (0, \infty)\) but that it does not converge uniformly on \([0, 1] \subset (0, \infty)\).

**11.10 Definition.**

The \textit{Laplace transform} of a function \(f : (0, \infty) \to \mathbb{R}\) is said to exist at a point \(s \in (0, \infty)\) if and only if the integral

\[ \mathcal{L}\{f\}(s) := \int_0^\infty e^{-st} f(t) \, dt \]

converges. (Note: This integral is improper at \(\infty\) and may be improper at 0.)

*11.1.9. Prove that

a) \(\mathcal{L}\{1\}(s) = \frac{1}{s}, \quad s > 0\)

b) \(\mathcal{L}\{t^n\}(s) = \frac{n!}{s^{n+1}}, \quad s > 0, \ n \in \mathbb{N}\)

c) \(\mathcal{L}\{e^{at}\}(s) = \frac{1}{s - a}, \quad s > a, \ a \in \mathbb{R}\)

d) \(\mathcal{L}\{\cos(bt)\}(s) = \frac{s}{s^2 + b^2}, \quad s > 0, \ b \in \mathbb{R}\)

e) \(\mathcal{L}\{\sin(bt)\}(s) = \frac{b}{s^2 + b^2}, \quad s > 0, \ b \in \mathbb{R}\)
11.1.10. Suppose that \( f : (0, \infty) \to \mathbb{R} \) is continuous and bounded and that \( \mathcal{L}(f) \) exists at some \( a \in (0, \infty) \). Let

\[
\phi(t) = \int_{0}^{t} e^{-au} f(u) \, du, \quad t \in (0, \infty).
\]

a) Prove that

\[
\int_{0}^{N} e^{-st} f(t) \, dt = \phi(N) e^{-(s-a)N} + (s-a) \int_{0}^{N} e^{-(s-a)t} \phi(t) \, dt
\]

for all \( N \in \mathbb{N} \).

b) Prove that the integral \( \int_{0}^{\infty} e^{-(s-a)t} \phi(t) \, dt \) converges uniformly on \( (b, \infty) \) for any \( b > a \) and

\[
\int_{0}^{\infty} e^{-st} f(t) \, dt = (s-a) \int_{0}^{\infty} e^{-(s-a)t} \phi(t) \, dt, \quad s > a.
\]

c) Prove that \( \mathcal{L}(f) \) exists, is continuous on \( (a, \infty) \), and satisfies

\[
\lim_{s \to \infty} \mathcal{L}(f)(s) = 0.
\]

d) Let \( g(t) = tf(t) \) for \( t \in (0, \infty) \). Prove that \( \mathcal{L}(f) \) is differentiable on \( (a, \infty) \) and

\[
\frac{d}{ds} \mathcal{L}(f)(s) = -\mathcal{L}(g)(s)
\]

for all \( s \in (a, \infty) \).

e) If, in addition, \( f' \) is continuous and bounded on \( (0, \infty) \), prove that

\[
\mathcal{L}(f')(s) = s \mathcal{L}(f)(s) - f(0)
\]

for all \( s \in (a, \infty) \).

11.1.11. Using Exercises 11.1.9 and 11.1.10, find the Laplace transforms of each of the functions \( tf^t \), \( t \sin \pi t \), and \( t^2 \cos t \).

11.2 THE DEFINITION OF DIFFERENTIABILITY

In this section we define what it means for a vector function to be differentiable at a point. Whatever our definition, we expect two things: If \( \mathbf{f} \) is differentiable at \( \mathbf{a} \), then \( \mathbf{f} \) will be continuous at \( \mathbf{a} \), and all first-order partial derivatives of \( \mathbf{f} \) will exist at \( \mathbf{a} \).

Working by analogy with the one-variable case, we guess that \( \mathbf{f} \) is differentiable at \( \mathbf{a} \) if and only if all its first-order partial derivatives exist at \( \mathbf{a} \). The following example shows that this guess is wrong even when the range of \( \mathbf{f} \) is one dimensional.
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**Proof.** If \((x, y) \neq (0, 0)\), then we can use the one-dimensional Product Rule to verify that both \(f_x\) and \(f_y\) exist and are continuous, for example,

\[
f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \cos \frac{1}{\sqrt{x^2 + y^2}} + 2x \sin \frac{1}{\sqrt{x^2 + y^2}}.
\]

Thus \(f\) is differentiable on \(\mathbb{R}^2 \setminus \{(0, 0)\}\). Since \(f_x(x, 0)\) has no limit as \(x \to 0\), the partial derivative \(f_x\) is not continuous at \((0, 0)\). A similar statement holds for \(f_y\). Thus to check differentiability at \((0, 0)\) we must return to the definition.

First, we compute the partial derivatives at \((0, 0)\). By definition,

\[
f_x(0, 0) = \lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = \lim_{t \to 0} t \sin \frac{1}{|t|} = 0,
\]

and similarly, \(f_y(0, 0) = 0\). Thus, both first partials exist at \((0, 0)\) and \(\nabla f(0, 0) = (0, 0)\).

To prove that \(f\) is differentiable at \((0, 0)\), we must verify (4) for \(a = (0, 0)\). But it is clear that

\[
\frac{f(h, k) - f(0, 0) - \nabla f(0, 0) \cdot (h, k)}{\|(h, k)\|} = \frac{\sqrt{h^2 + k^2} \sin \frac{1}{\sqrt{h^2 + k^2}}}{\sqrt{h^2 + k^2}} \to 0
\]

as \((h, k) \to (0, 0)\). Thus \(f\) is differentiable at \((0, 0)\).

**EXERCISES**

11.2.1. Suppose, for \(j = 1, 2, \ldots, n\), that \(f_j\) are real functions continuously differentiable on the interval \((-1, 1)\). Prove that

\[
g(x) := f_1(x_1) \cdots f_n(x_n)
\]

is differentiable on the cube \((-1, 1) \times (-1, 1) \times \cdots \times (-1, 1)\).

11.2.2. Suppose that \(f, g : \mathbb{R} \to \mathbb{R}^m\) are differentiable at \(a\) and there is a \(\delta > 0\) such that \(g(x) \neq 0\) for all \(0 < |x - a| < \delta\). If \(f(a) = g(a) = 0\) and \(Dg(a) \neq 0\), prove that

\[
\lim_{x \to a} \frac{\|f(x)\|}{\|g(x)\|} = \frac{\|Df(a)\|}{\|Dg(a)\|}
\]

11.2.3. Prove that \(f(x, y) = \sqrt{|xy|}\) is not differentiable at \((0, 0)\).

11.2.4. Prove that

\[
f(x, y) = \begin{cases} 
\frac{x^2 + y^2}{\sin \sqrt{x^2 + y^2}} & 0 < \|(x, y)\| < \pi \\
0 & (x, y) = (0, 0)
\end{cases}
\]

is not differentiable at \((0, 0)\).
11.2.5. Prove that

\[ f(x, y) = \begin{cases} \frac{x^4 + y^4}{(x^2 + y^2)\alpha} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

is differentiable on \( \mathbb{R}^2 \) for all \( \alpha < 3/2 \).

11.2.6. Prove that if \( \alpha > 1/2 \), then

\[ f(x, y) = \begin{cases} \frac{|xy|^\alpha \log(x^2 + y^2)}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

is differentiable at \((0, 0)\).

11.2.7. Prove that

\[ f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases} \]

is continuous on \( \mathbb{R}^2 \) and has first-order partial derivatives everywhere on \( \mathbb{R}^2 \), but \( f \) is not differentiable at \((0, 0)\).

**11.2.8.** This exercise is used several times in this chapter and the next. Suppose that \( T \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m) \). Prove that \( T \) is differentiable everywhere on \( \mathbb{R}^n \) with \( DT(a) = T \) for \( a \in \mathbb{R}^n \).

11.2.9. Let \( r > 0 \), \( f : B_r(0) \to \mathbb{R} \), and suppose that there exists an \( \alpha > 1 \) such that \( |f(x)| \leq \|x\|^\alpha \) for all \( x \in B_r(0) \). Prove that \( f \) is differentiable at \( 0 \).

11.2.10. Let \( V \) be open in \( \mathbb{R}^n \), \( a \in V \), and \( f : V \to \mathbb{R}^m \).

**11.19 Definition.**

If \( u \) is a unit vector in \( \mathbb{R}^n \) (i.e., \( \|u\| = 1 \)), then the **directional derivative** of \( f \) at \( a \) in the direction \( u \) is defined by

\[ D_u f(a) := \lim_{t \to 0} \frac{f(a + tu) - f(a)}{t} \]

when this limit exists.

a) Prove that \( D_u f(a) \) exists for \( u = e_k \) if and only if \( f_{x_k}(a) \) exists, in which case

\[ D_{e_k} f(a) = \frac{\partial f}{\partial x_k}(a). \]

b) Show that if \( f \) has directional derivatives at \( a \) in all directions \( u \), then the first-order partial derivatives of \( f \) exist at \( a \). In fact, use Example 11.11 to show that the converse of this statement is false.

11.3 **DERIVATIVES, DIFFERENTIALS, AND GRADIENTS.**

In this section, we examine how functions change.

11.20 **Theorem:** Let \( f \) and \( g \) be differentiable at \( a \). In fact, use Example 11.11 to show that the converse of this statement is false.

[The sums of... multiplication... ]
c) Prove that the directional derivatives of

\[ f(x, y) = \begin{cases} 
\frac{x^2 y}{x^2 + y^2} & (x, y) \neq (0, 0) \\
0 & (x, y) = (0, 0)
\end{cases} \]

exist at (0, 0) in all directions \( \mathbf{u} \), but \( f \) is neither continuous nor differentiable at (0, 0).

11.2.11. Let \( r > 0 \), \((a, b) \in \mathbb{R}^2\), \( f : B_r(a, b) \to \mathbb{R} \), and suppose that the first-order partial derivatives \( f_x \) and \( f_y \) exist in \( B_r(a, b) \) and are differentiable at \((a, b)\).

a) Set \( \Delta(h) = f(a + h, b + h) - f(a + h, b) - f(a, b + h) + f(a, b) \) and prove for \( h \) sufficiently small that

\[
\frac{\Delta(h)}{h} = f_y(a + h, b + th) - f_y(a, b) - \nabla f_y(a, b) \cdot (h, th) \\
- \left( f_y(a, b + th) - f_y(a, b) - \nabla f_y(a, b) \cdot (0, th) \right) + hf_{yx}(a, b)
\]

for some \( t \in (0, 1) \).

b) Prove that

\[
\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b).
\]

c) Prove that

\[
\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b).
\]

11.3 DERIVATIVES, DIFFERENTIALS, AND TANGENT PLANES

In this section we begin to explore the analogy between \( Df \) and \( f' \). First we examine how the total derivative interacts with the algebra of functions.

11.20 Theorem. Let \( \alpha \in \mathbb{R} \), \( \mathbf{a} \in \mathbb{R}^n \), and suppose that \( \mathbf{f} \) and \( \mathbf{g} \) are vector functions. If \( \mathbf{f} \) and \( \mathbf{g} \) are differentiable at \( \mathbf{a} \), then \( \mathbf{f} + \mathbf{g} \), \( \alpha \mathbf{f} \), and \( \mathbf{f} \cdot \mathbf{g} \) are all differentiable at \( \mathbf{a} \). In fact,

\[
D(\mathbf{f} + \mathbf{g})(\mathbf{a}) = D\mathbf{f}(\mathbf{a}) + D\mathbf{g}(\mathbf{a}), \quad (7)
\]

\[
D(\alpha \mathbf{f})(\mathbf{a}) = \alpha D\mathbf{f}(\mathbf{a}), \quad (8)
\]

and

\[
D(\mathbf{f} \cdot \mathbf{g})(\mathbf{a}) = \mathbf{g}(\mathbf{a})D\mathbf{f}(\mathbf{a}) + \mathbf{f}(\mathbf{a})D\mathbf{g}(\mathbf{a}). \quad (9)
\]

[The sums which appear on the right side of (7) and (9) represent matrix addition, and the products which appear on the right side of (9) represent matrix multiplication.]
Since \( Q \to 0 \) as \( h \to 0 \), we conclude by the Squeeze Theorem that 
\( \varepsilon(h)/\|h\| \to 0 \) as \( h \to 0 \). In particular, \( f \) is differentiable at \((a, b)\) and 
\((f_x(a, b), f_y(a, b), -1)\) is normal to its tangent plane there.

**EXERCISES**

11.3.1. For each of the following, prove that \( f \) and \( g \) are differentiable on their domains, and find formulas for \( D(f + g)(x) \) and \( D(f \cdot g)(x) \).

a) \[ f(x, y) = x - y, \quad g(x, y) = x^2 + y^2 \]
b) \[ f(x, y) = xy, \quad g(x, y) = x \sin x - \cos y \]
c) \[ f(x, y) = (\cos(xy), x \log y), \quad g(x, y) = (y, x) \]
d) \[ f(x, y, z) = (y, x - z), \quad g(x, y, z) = (xyz, y^2) \]

11.3.2. For each of the following functions, find an equation of the tangent plane to \( z = f(x, y) \) at \( c \).

a) \[ f(x, y) = x^2 + y^2, \quad c = (1, -1, 2) \]
b) \[ f(x, y) = x^3 y - xy^3, \quad c = (1, 1, 0) \]
c) \[ f(x, y, z) = xy + \sin z, \quad c = (1, 0, \pi/2, 1) \]

11.3.3. Find all points on the paraboloid \( z = x^2 + y^2 \) (see Appendix D) where the tangent plane is parallel to the plane \( x + y + z = 1 \). Find equations of the corresponding tangent planes. Sketch the graphs of these functions to see that your answer agrees with your intuition.

11.3.4. Let \( \mathcal{K} \) be the cone, given by \( z = \sqrt{x^2 + y^2} \).

a) Find an equation of each plane tangent to \( \mathcal{K} \) which is perpendicular to the plane \( x + z = 5 \).

b) Find an equation of each plane tangent to \( \mathcal{K} \) which is parallel to the plane \( x - y + z = 1 \).

11.3.5. Prove (7) and (8) in Theorem 11.20.

11.3.6. [QUOTIENT RULE]. Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable at \( a \) and that \( f(a) \neq 0 \).

a) Show that for \( \|h\| \) sufficiently small, \( f(a + h) \neq 0 \).

b) Prove that \( Df(a)(h)/\|h\| \) is bounded for all \( h \in \mathbb{R}^n \setminus \{0\} \).

c) If \( T := -Df(a)/f^2(a) \), show that

\[
\frac{1}{f(a + h)} - \frac{1}{f(a)} - T(h) = \frac{f(a) - f(a + h) + Df(a)(h)}{f(a)f(a + h)} + \frac{(f(a + h) - f(a))Df(a)(h)}{f^2(a)f(a + h)}
\]

for \( \|h\| \) sufficiently small.
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d) Prove that \( 1/f(x) \) is differentiable at \( x = a \) and

\[
D \left( \frac{1}{f} \right) (a) = -\frac{Df(a)}{f^2(a)}.
\]

e) Prove that if \( f \) and \( g \) are real-valued vector functions which are
differentiable at some \( a \), and if \( g(a) \neq 0 \), then

\[
D \left( \frac{f}{g} \right) (a) = \frac{g(a)Df(a) - f(a)Dg(a)}{g^2(a)}.
\]

11.3.7. [Cross-Product Rule]. Suppose that \( V \) is open in \( \mathbb{R}^n \), that \( f, g : V \rightarrow \mathbb{R}^3 \), and that \( a \in V \). If \( f \) and \( g \) are differentiable at \( a \), prove that \( f \times g \) is
differentiable at \( a \) and

\[
D(f \times g)(a)(y) = f(a) \times (Dg(a)(y)) - g(a) \times (Df(a)(y))
\]

for all \( y \in \mathbb{R}^n \).

*11.3.8. Compute the differential of each of the following functions.

a) \( z = x^2 + y^2 \)  

b) \( z = \sin(xy) \)  

c) \( z = \frac{xy}{1 + x^2 + y^2} \)

*11.3.9. Let \( w = x^2y + z \). Use differentials to approximate \( \Delta w \) as \( (x, y, z) \) moves from (1,2,1) to (1.01, 1.98, 1.03). Compare your approximation

with the actual value of \( \Delta w \).

*11.3.10. The time \( T \) it takes for a pendulum to complete one full swing is
given by

\[
T = 2\pi \sqrt{\frac{L}{g}}
\]

where \( g \) is the acceleration due to gravity and \( L \) is the length of the
pendulum. If \( g \) can be measured with a maximum error of 1%, how
accurately must \( L \) be measured (in terms of percentage error) so that
the calculated value of \( T \) has a maximum error of 2%?

*11.3.11. Suppose that

\[
\frac{1}{w} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},
\]

where each variable \( x, y, z \) can be measured with a maximum error of
\( p\% \). Prove that the calculated value of \( w \) also has a maximum error
of \( p\% \).

11.4 THE CHAIN RULE

Here is the Chain Rule for vector functions.

11.28 Theorem

Suppose \( T(\mathbf{h}) \) and \( f(b) \) are \( p \times m \) matrices for total derivates. We show that

\[
T(\mathbf{h}) = f(b)
\]

Set

\[
\mathbf{h} \rightarrow 0 \quad \text{in} \quad \mathbb{R}^n
\]

for \( ||\mathbf{h}|| < \delta \) and \( \mathbf{k} = g(a - b) \). Hence, we have

It remains to show that

Since it is clear that \( \mathbf{h} \rightarrow 0 \) as \( \mathbf{k} \rightarrow 0 \) in the triangle

\[
||\mathbf{k}|| = ||\mathbf{h}||
\]

Thus \( ||\mathbf{k} || = ||\mathbf{h}|| \), so that \( \mathbf{k} \rightarrow 0 \), and it follows that

\[
Df(g(a - b)) = T(\mathbf{h}) - f(b)
\]

as \( \mathbf{h} \rightarrow 0 \).
The Chain Rule can be used to compute individual partial derivatives without writing out the entire matrices $Df$ and $Dg$. For example, suppose that $f(u_1, \ldots, u_m)$ is differentiable from $\mathbb{R}^m$ to $\mathbb{R}$, that $g(x_1, \ldots, x_n)$ is differentiable from $\mathbb{R}^n$ to $\mathbb{R}^m$, and that $z = f(g(x_1, \ldots, x_n))$. Since $Df = \nabla f$ and the $j$th column of $Dg$ consists of first partial derivatives, with respect to $x_j$, of the components $u_k := g_k(x_1, \ldots, x_n)$, it follows from the Chain Rule and the definition of matrix multiplication that

$$\frac{\partial z}{\partial x_j} = \frac{\partial f}{\partial u_1} \frac{\partial u_1}{\partial x_j} + \cdots + \frac{\partial f}{\partial u_m} \frac{\partial u_m}{\partial x_j}$$

for $j = 1, 2, \ldots, n$. Here are two concrete examples which illustrate this principle.

11.29 EXAMPLES.

i) If $F, G, H : \mathbb{R}^2 \to \mathbb{R}$ are differentiable and $z = F(x, y)$, where $x = G(r, \theta)$, and $y = H(r, \theta)$, then

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}$$

and

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}$$

ii) If $f : \mathbb{R}^3 \to \mathbb{R}$ and $\phi, \psi, \sigma : \mathbb{R} \to \mathbb{R}$ are differentiable and $w = f(x, y, z)$, where $x = \phi(t)$, $y = \psi(t)$, and $z = \sigma(t)$, then

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}.$$ 

EXERCISES

11.4.1. Let $F : \mathbb{R}^3 \to \mathbb{R}$ and $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ be $C^2$ functions. If $w = F(x, y, z)$, where $x = f(p, q)$, $y = g(p, q)$, and $z = h(p, q)$, find formulas for $w_p, w_q,$ and $w_{pq}.

11.4.2. Let $r > 0$, let $a \in \mathbb{R}^n$, and suppose that $g : B_r(a) \to \mathbb{R}^n$ is differentiable at $a$.

a) If $f : B_r(g(a)) \to \mathbb{R}$ is differentiable at $g(a)$, prove that the partial derivatives of $h = f \circ g$ are given by

$$\frac{\partial h}{\partial x_j}(a) = \nabla f(g(a)) \cdot \frac{\partial g}{\partial x_j}(a)$$

for $j = 1, 2, \ldots, n$.

b) If $n = m$ and $f : B_r(g(a)) \to \mathbb{R}^n$ is differentiable at $g(a)$, prove that

$$\det(D(f \circ g)(a)) = \det(Df(g(a))) \det(Dg(a)).$$
11.4.3. Suppose that $k \in \mathbb{N}$ and that $f : \mathbb{R}^n \to \mathbb{R}$ is homogeneous of order $k$; that is, that $f(\rho x) = \rho^k f(x)$ for all $x \in \mathbb{R}^n$ and all $\rho \in \mathbb{R}$. If $f$ is differentiable on $\mathbb{R}^n$, prove that

$$x_1 \frac{\partial f}{\partial x_1}(x) + \cdots + x_n \frac{\partial f}{\partial x_n}(x) = kf(x)$$

for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

11.4.4. Let $f, g : \mathbb{R} \to \mathbb{R}$ be twice differentiable. Prove that $u(x, y) := f(xy)$ satisfies

$$x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} = 0,$$

and $v(x, y) := f(x - y) + g(x + y)$ satisfies the wave equation; that is,

$$\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = 0.$$

11.4.5. Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and satisfy the Cauchy–Riemann equations; that is, that

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad \text{and} \quad \frac{\partial f}{\partial y} = -\frac{\partial g}{\partial x}$$

hold on $\mathbb{R}^2$. If $u(r, \theta) = f(r \cos \theta, r \sin \theta)$ and $v(r, \theta) = g(r \cos \theta, r \sin \theta)$, prove that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta} \quad r \neq 0.$$

11.4.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $C^2$ on $\mathbb{R}^2$ and set $u(r, \theta) = f(r \cos \theta, r \sin \theta)$. If $f$ satisfies the Laplace equation; that is, if

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

prove for each $r \neq 0$ that

$$\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial r^2} = 0.$$

11.4.7. Let

$$u(x, t) = \frac{e^{-x^2/4t}}{\sqrt{4\pi t}}, \quad t > 0, \ x \in \mathbb{R}.$$

a) Prove that $u$ satisfies the heat equation (i.e., $u_{xx} - u_t = 0$ for all $t > 0$ and $x \in \mathbb{R}$).
b) If \( a > 0 \), prove that \( u(x, t) \to 0 \), as \( t \to 0^+ \), uniformly for \( x \in [a, \infty) \).

11.4.8. Let \( u : \mathbb{R} \to [0, \infty) \) be differentiable. Prove that for each \((x, y, z) \neq (0, 0, 0)\),

\[
F(x, y, z) := u \left( \sqrt{x^2 + y^2 + z^2} \right)
\]

satisfies

\[
\left( \left( \frac{\partial F}{\partial x} \right)^2 + \left( \frac{\partial F}{\partial y} \right)^2 + \left( \frac{\partial F}{\partial z} \right)^2 \right)^{1/2} = u' \left( \sqrt{x^2 + y^2 + z^2} \right).
\]

11.4.9. Suppose that \( z = F(x, y) \) is differentiable at \((a, b)\), that \( F_z(a, b) \neq 0 \), and that \( I \) is an open interval containing \( a \). Prove that if \( f : I \to \mathbb{R} \) is differentiable at \( a \), \( f(a) = b \), and \( F(x, f(x)) = 0 \) for all \( x \in I \), then

\[
\frac{df}{dx}(a) = \frac{-\frac{\partial F}{\partial x}(a, b)}{\frac{\partial F}{\partial y}(a, b)}.
\]

11.4.10. Suppose that \( I \) is a nonempty, open interval and that \( f : I \to \mathbb{R}^n \) is differentiable on \( I \). If \( f(t) \subseteq \partial B_r(0) \) for some fixed \( r > 0 \), prove that \( f(t) \) is orthogonal to \( f'(t) \) for all \( t \in I \).

11.4.11. Let \( V \) be open in \( \mathbb{R}^n \), \( a \in V \), \( f : V \to \mathbb{R} \), and suppose that \( f \) is differentiable at \( a \).

a) Prove that the directional derivative \( D_uf(a) \) exists (see Exercise 11.2.10) for each \( u \in \mathbb{R}^n \) such that \( ||u|| = 1 \) and \( D_uf(a) = \nabla f(a) \cdot u \).

b) If \( \nabla f(a) \neq 0 \) and \( \theta \) represents the angle between \( u \) and \( \nabla f(a) \), prove that \( D_uf(a) = ||\nabla f(a)|| \cos \theta \).

c) Show that as \( u \) ranges over all unit vectors in \( \mathbb{R}^n \), the maximum of \( D_uf(a) \) is \( ||\nabla f(a)|| \), and it occurs when \( u \) is parallel to \( \nabla f(a) \).

11.5 THE MEAN VALUE THEOREM AND TAYLOR'S FORMULA

Using \( Df \) as a replacement for \( f' \), we guess that the multidimensional analogue of the Mean Value Theorem is \( f(x) - f(a) = Df(c)(x - a) \) for some \( c \) "between" \( x \) and \( a \); that is, some \( c \in L(x; a) \), the line segment from \( a \) to \( x \). The following result shows that this guess is correct when \( f \) is real valued (see also Exercises 11.5.6 and 11.5.9).

11.30 Theorem. [MEAN VALUE THEOREM FOR REAL VALUED FUNCTIONS].

Let \( V \) be open in \( \mathbb{R}^n \) and suppose that \( f : V \to \mathbb{R} \) is differentiable on \( V \). If \( x, a \in V \) and \( L(x; a) \subseteq V \), then there is a \( c \in L(x; a) \) such that

\[
f(x) - f(a) = \nabla f(c) \cdot (x - a).
\]

(24)
Solution. It is easy to verify that $f_x, f_y, f_{xx}, f_{xy},$ and $f_{yy}$ are all zero at $(0,0)$, so $D^{(1)} f((0,0); (x, y)) = 0$ and $D^{(2)} f((0,0); (x, y)) = 0$. Since $f_x(x, y) = y^3 \sin(xy), f_{xx} = -2y \cos(xy) + x^2 \sin(xy), f_{xy} = x^3 \cos(xy) + x^2 \sin(xy),$ and $f_{yy} = x^3 \cos(xy),$, Theorem 11.2 implies $D^{(3)} f((c, d); (x, y)) = (x^3 + y^3) \sin(cd) + 3(x^2y + xy^2) \sin(cd) - 6(x + y) \cos(cd)$. Thus by Taylor's Formula, for all $(x, y) \in \mathbb{R}^2$ there is a point $(c, d)$ on the line segment between $(0,0)$ and $(x, y)$ such that

$$
\cos(xy) = 1 + \left(\frac{x^3 + y^3}{6}\right) \sin(cd) + \left(\frac{x^2y + xy^2}{2}\right) \sin(cd) - (x + y) \cos(cd).
$$

EXERCISES

11.5.1. a) Write out an expression in powers of $(x+1)$ and $(y-1)$ for $f(x, y) = x^2 + xy + y^2$.

b) Write Taylor's Formula for $f(x, y) = \sqrt{x} + \sqrt{y}, a = (1, 4)$, and $p = 3$.

c) Write Taylor's Formula for $f(x, y) = e^{xy}, a = (0, 0)$, and $p = 4$.

11.5.2. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^p$ on $B_r(x_0, y_0)$ for some $r > 0$. Prove that, given $(x, y) \in B_r(x_0, y_0)$, there is a point $(c, d)$ on the line segment between $(x, y)$ and $(x_0, y_0)$ such that

$$
f(x, y) = f(x_0, y_0) + \sum_{k=1}^{p-1} \frac{1}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (x - x_0)^j (y - y_0)^{k-j} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(x_0, y_0)\right) + \frac{1}{p!} \sum_{j=0}^{p} \binom{p}{j} (x - x_0)^j (y - y_0)^{p-j} \frac{\partial^p f}{\partial x^j \partial y^{p-j}}(c, d).
$$

11.5.3. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are differentiable on $\mathbb{R}^n$ and that there exist $r > 0$ and $a \in \mathbb{R}^n$ such that $Dg(x)$ is the identity matrix, $I$, for all $x \in B_r(a)$. Prove that there is a function $h : B_r(a) \setminus \{a\} \to B_r(x)$ such that

$$
\frac{|f(g(x)) - f(g(a))|}{\|x - a\|} \leq \|Df((g \circ h)(x))\|
$$

for all $x \in B_r(a) \setminus \{a\}$.

11.5.4. Suppose that $V$ is convex and open in $\mathbb{R}^n$ and that $f : V \to \mathbb{R}^n$ is differentiable on $V$. If there exists an $a \in V$ such that $Df(x) = Df(a)$ for all $x \in V$, prove that there exist a linear function $S \in L(\mathbb{R}^n; \mathbb{R}^n)$ and a vector $c \in \mathbb{R}^n$ such that $f(x) = S(x) + c$ for all $x \in V$.

11.5.5. [INTEGRAL FORM OF TAYLOR'S FORMULA]. Let $p \in \mathbb{N}, V$ be an open set in $\mathbb{R}^n, x, a \in V$, and $f : V \to \mathbb{R}$ be $C^p$ on $V$. If $L(x; a) \subset V$ and $h = x - a$, prove that

$$
\int_{L(x; a)} f^{(p)}(x) \, dx = \int_{L(x; a)} f^{(p-1)}(x) \, dx.
$$

11.5.6. Let $B_r(x_0, y_0)$ with $r > 0$. Let $f(x, y) = x^2 + y^2$.

a) Find a point $(c, d)$ on the line segment between $(0,0)$ and $(x, y)$ such that

$$
f(x, y) = f(0,0) + \sum_{k=1}^{p} \frac{1}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (x - 0)^j (y - 0)^{k-j} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(0,0)\right).
$$

b) Write Taylor's Formula for $f(x, y) = e^{xy}, a = (0, 0)$, and $p = 4$.

11.5.7. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^p$ on $B_r(x_0, y_0)$ for some $r > 0$. Prove that there is a point $(c, d)$ on the line segment between $(x, y)$ and $(x_0, y_0)$ such that

$$
f(x, y) = f(x_0, y_0) + \sum_{k=1}^{p-1} \frac{1}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (x - x_0)^j (y - y_0)^{k-j} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(x_0, y_0)\right) + \frac{1}{p!} \sum_{j=0}^{p} \binom{p}{j} (x - x_0)^j (y - y_0)^{p-j} \frac{\partial^p f}{\partial x^j \partial y^{p-j}}(c, d).
$$

11.5.8. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^p$ on $B_r(x_0, y_0)$ for some $r > 0$. Prove that there is a point $(c, d)$ on the line segment between $(x, y)$ and $(x_0, y_0)$ such that

$$
f(x, y) = f(x_0, y_0) + \sum_{k=1}^{p} \frac{1}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (x - x_0)^j (y - y_0)^{k-j} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(x_0, y_0)\right) + \frac{1}{p!} \sum_{j=0}^{p} \binom{p}{j} (x - x_0)^j (y - y_0)^{p-j} \frac{\partial^p f}{\partial x^j \partial y^{p-j}}(c, d).
$$

11.5.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$, and let $B_r(x_0, y_0)$ with $r > 0$. Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ is $C^p$ on $B_r(x_0, y_0)$ for some $r > 0$. Prove that there is a point $(c, d)$ on the line segment between $(x, y)$ and $(x_0, y_0)$ such that

$$
f(x, y) = f(x_0, y_0) + \sum_{k=1}^{p} \frac{1}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (x - x_0)^j (y - y_0)^{k-j} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(x_0, y_0)\right) + \frac{1}{p!} \sum_{j=0}^{p} \binom{p}{j} (x - x_0)^j (y - y_0)^{p-j} \frac{\partial^p f}{\partial x^j \partial y^{p-j}}(c, d).
$$

11.5.10. Let $f : \mathbb{R}^n \to \mathbb{R}$, and let $B_r(x_0, y_0)$ with $r > 0$. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is $C^p$ on $B_r(x_0, y_0)$ for some $r > 0$. Prove that there is a point $(c, d)$ on the line segment between $(x, y)$ and $(x_0, y_0)$ such that

$$
f(x, y) = f(x_0, y_0) + \sum_{k=1}^{p} \frac{1}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (x - x_0)^j (y - y_0)^{k-j} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(x_0, y_0)\right) + \frac{1}{p!} \sum_{j=0}^{p} \binom{p}{j} (x - x_0)^j (y - y_0)^{p-j} \frac{\partial^p f}{\partial x^j \partial y^{p-j}}(c, d).
$$

11.5.11. Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is $C^p$ on $B_r(x_0, y_0)$ for some $r > 0$. Prove that

$$
f(x, y) = f(x_0, y_0) + \sum_{k=1}^{p} \frac{1}{k!} \left(\sum_{j=0}^{k} \binom{k}{j} (x - x_0)^j (y - y_0)^{k-j} \frac{\partial^k f}{\partial x^j \partial y^{k-j}}(x_0, y_0)\right) + \frac{1}{p!} \sum_{j=0}^{p} \binom{p}{j} (x - x_0)^j (y - y_0)^{p-j} \frac{\partial^p f}{\partial x^j \partial y^{p-j}}(c, d).
$$
\[ f(x) - f(a) = \sum_{k=1}^{p-1} \frac{1}{k!} D^{(k)} f(a; h) + \frac{1}{(p-1)!} \int_0^1 (1-t)^{p-1} D^{(p)} f(a + rh; h) \, dr. \]

11.5.6. Let \( r > 0 \), \( a, b \in \mathbb{R} \), \( f : B_r(a, b) \to \mathbb{R} \) be differentiable, and \((x, y) \in B_r(a, b)\).

a) Let \( g(t) = f(tx + (1-t)a, y) + f(a, ty + (1-t)b) \) and compute the derivative of \( g \).

b) Prove that there are numbers \( c \) between \( a \) and \( x \), and \( d \) between \( b \) and \( y \) such that

\[ f(x, y) - f(a, b) = (x-a)f_x(c, y) + (y-b)f_y(a, d). \]

(This is Exercise 12.20 in Apostol [1].)

11.5.7. Suppose that \( 0 < r < 1 \) and that \( f : B_r(0) \to \mathbb{R} \) is continuously differentiable. If there is an \( \alpha > 0 \) such that \( |f(x)| \leq ||x||^\alpha \) for all \( x \in B_r(0) \), prove that there is an \( M > 0 \) such that \( |f(x)| \leq M||x|| \) for \( x \in B_r(0) \).

11.5.8. Suppose that \( V \) is open in \( \mathbb{R}^n \), that \( f : V \to \mathbb{R} \) is \( C^2 \) on \( V \), and that \( f_{x_j}(a) = 0 \) for some \( a \in H \) and all \( j = 1, \ldots, n \). Prove that if \( H \) is a compact convex subset of \( V \), then there is a constant \( M \) such that

\[ |f(x) - f(a)| \leq M||x - a||^2 \]

for all \( x \in H \).

11.5.9. Let \( f : \mathbb{R}^n \to \mathbb{R} \). Suppose that for each unit vector \( u \in \mathbb{R}^n \), the directional derivative \( D_u f(a + tu) \) exists for \( t \in [0, 1] \) (see Definition 11.19). Prove that

\[ f(a + u) - f(a) = D_u f (a + tu) \]

for some \( t \in (0, 1) \).

11.5.10. Suppose that \( V \) is open in \( \mathbb{R}^2 \), that \((a, b) \in V \), and that \( f : V \to \mathbb{R} \) is \( C^3 \) on \( V \). Prove that

\[ \lim_{r \to 0} \frac{4}{\pi r^2} \int_0^{2\pi} f(a + r \cos \theta, b + r \sin \theta) \cos(2\theta) \, d\theta = f_{xx}(a, b) - f_{xy}(a, b). \]

11.5.11. Suppose that \( V \) is open in \( \mathbb{R}^2 \), that \( H = [a, b] \times [0, c] \subset V \), that \( u : V \to \mathbb{R} \) is \( C^2 \) on \( V \), and that \( u(x_0, t_0) \geq 0 \) for all \((x_0, t_0) \in \partial H \).

a) Show that, given \( \varepsilon > 0 \), there is a compact set \( K \subset H^\circ \) such that \( u(x, t) \geq -\varepsilon \) for all \((x, t) \in H \setminus K \).

b) Suppose that \( u(x_1, t_1) = -\ell < 0 \) for some \((x_1, t_1) \in H^\circ \), and choose \( r > 0 \) so small that \( 2rt_1 < \ell \). Apply part a) to \( \varepsilon := \ell/2 - rt_1 \) to choose the compact set \( K \), and prove that the minimum of \( w(x, t) := u(x, t) + r(t - t_1) \) on \( H \) occurs at some \((x_2, t_2) \in K \).
c) Prove that if \( u \) satisfies the heat equation (i.e., \( u_{xx} - u_t = 0 \) on \( V \)), and if \( u(x_0, t_0) \geq 0 \) for all \((x_0, t_0) \in \partial H\), then \( u(x, t) \geq 0 \) for all \((x, t) \in H\).

11.5.12. a) Prove that every convex set in \( \mathbb{R}^n \) is connected.
   b) Show that the converse of part a) is false.
   *c) Suppose that \( f : \mathbb{R} \to \mathbb{R} \). Prove that \( f \) is convex (as a function) if and only if \( E := \{(x, y) : y \geq f(x)\} \) is convex (as a set in \( \mathbb{R}^2 \)).

11.6 THE INVERSE FUNCTION THEOREM

By the one-dimensional Inverse Function Theorem (Theorem 4.33), if \( g : \mathbb{R} \to \mathbb{R} \) is 1-1 and differentiable with \( g'(x_0) \neq 0 \), then \( g^{-1} \) is differentiable at \( y_0 = g(x_0) \) and

\[
(g^{-1})'(y_0) = \frac{1}{g'(x_0)}.
\]

In this section we obtain a multivariable analogue of this result (i.e., an Inverse Function Theorem for vector functions \( f \) from \( n \) variables to \( n \) variables). What shall we use for hypotheses? We needed \( g \) to be 1-1 so that the inverse function \( g^{-1} \) existed. For the same reason, we shall assume that \( f \) is 1-1. We needed \( g'(x_0) \) to be nonzero so that we could divide by it. In the multidimensional case, \( Df(a) \) is a matrix; hence “divisibility” corresponds to invertibility. Since an \( n \times n \) matrix is invertible if and only if it has a nonzero determinant (see Appendix C), we shall assume that the Jacobian of \( f \)

\[
\Delta f(a) := \det(Df(a)) \neq 0.
\]

The word Jacobian is used because it was Jacobi who first recognized the importance of \( \Delta f \) and its connection with volume (see Exercise 12.4.6).

The proof of the Inverse Function Theorem on \( \mathbb{R}^n \) is not simple and lies somewhat deeper than the preceding results of this chapter. Before presenting it, we first prove two preliminary results which explore the consequences of the hypothesis \( \Delta f \neq 0 \).

11.39 Lemma.

Let \( V \) be open and nonempty in \( \mathbb{R}^n \) and let \( f : V \to \mathbb{R}^n \) be continuous. If \( f \) is 1-1 and has first-order partial derivatives on \( V \), and if \( \Delta f \neq 0 \) on \( V \), then \( f^{-1} \) is continuous on \( f(V) \).

**Strategy:** To prove that \( f^{-1} \) is continuous on \( V \), it suffices to prove (apply Exercise 9.4.3 or Theorem 10.58 to \( f^{-1} \)) that \( f(W) = (f^{-1})^{-1}(W) \) is open for all open \( W \subseteq V \). Thus, given \( b \in f(W) \), say \( b = f(a) \) for some \( a \in W \), we must find a \( \rho > 0 \) such that \( B_r(a) \subseteq f(W) \). We will actually show more: that if \( B_r(a) \subseteq W \) for some \( r > 0 \), then there is a \( \rho > 0 \) such that \( B_r(b) \subseteq f(B_r(a)) \); that is, \( y \in B_r(b) \) implies that \( y = f(c) \) for some \( c \in B_r(a) \).

Where should we look to find such a \( c \)? To show that \( f(c) = y \), we might first try finding a point \( c \in B_r(a) \) that minimizes \( \|f(x) - y\| \) as \( x \) ranges over \( B_r(a) \) and then try \( f(c) \) as a candidate. Moreover, this approach works because of the inverse function theorem; it requires not the inverse function theorem but its interior normal form.

**Proof.** Set \( c = f^{-1}(y) \), then

\[
B_r(c) \subseteq f(B_r(a)),
\]

and then it is easy to prove the result. Moreover, the normal form of the inverse function theorem must be observed in the interior normal form.

Since \( f \) is continuous,

\[
\rho > 0
\]

is positive, and hence attained on \( \partial B_r(a) \); this completes the proof.

Set \( \rho \) be the radius of the smallest ball around \( c \) contained in \( B_r(c) \). Then \( h(a) = \|f(a) - y\| > 0 \) for all \( x \in B_r(a) \).

To show that \( h(a) \) is continuous on \( B_r(a) \), let \( \rho > h(c) \) for arbitrarily small \( \rho > h(c) \).

a contraction mapping.

It remains to show that \( f \) is continuous at \( c \).

for \( k = 1 \).
Then \( F(4, 3, 2, 1, -1, -2) = (0, 0) \), and
\[
\frac{\partial(F_1, F_2)}{\partial(u, v)} = \det \begin{bmatrix} 2u & 2v \\ 2u/x^2 & 2v/y^2 \end{bmatrix} = 4uv \left( \frac{1}{y^2} - \frac{1}{x^2} \right).
\]
This determinant is nonzero when \( u = 4, v = 3, x = 2, \) and \( y = 1 \). Therefore, such functions \( u, v \) exist by the Implicit Function Theorem.

**EXERCISES**

**11.6.1.** For each of the following functions, prove that \( f^{-1} \) exists and is differentiable in some nonempty, open set containing \((a, b)\), and compute \( D(f^{-1})(a, b)\)

a) \( f(u, v) = (3u - v, 2u + 5v) \) at any \((a, b) \in \mathbb{R}^2\)

b) \( f(u, v) = (u + v, \sin u + \cos v) \) at \((a, b) = (0, 1)\)

c) \( f(u, v) = (uv, u^2 + v^2) \) at \((a, b) = (2, 5)\)

d) \( f(u, v) = (u^2 - v^2, \sin u - \log v) \) at \((a, b) = (-1, 0)\)

**11.6.2.** For each of the following functions, find out whether the given expression can be solved for \( z \) in a nonempty, open set \( V \) containing \((0, 0, 0)\). Is the solution differentiable near \((0, 0, 0)\)?

a) \( xyz + \sin(x + y + z) = 0 \)

b) \( x^2 + y^2 + z^2 + \sqrt{\sin(x^2 + y^2)} + 3z + 4 = 2 \)

c) \( xyz(2 \cos y - \cos z) + (z \cos x - x \cos y) = 0 \)

d) \( x + y + z + g(x, y, z) = 0 \), where \( g \) is any continuously differentiable function which satisfies \( g(0, 0, 0) = 0 \) and \( g_z(0, 0, 0) > 0 \)

**11.6.3.** Prove that there exist functions \( u(x, y), v(x, y), \) and \( w(x, y) \), and an \( r > 0 \) such that \( u, v, w \) are continuously differentiable and satisfy the equations

\[
\begin{align*}
    u^5 + x^2y - y + w &= 0 \\
    v^5 + yu^2 - x + w &= 0 \\
    w^4 + y^5 - x^4 &= 1
\end{align*}
\]
on \( B_r(1, 1) \), and \( u(1, 1) = 1, v(1, 1) = 1, w(1, 1) = -1. \)

**11.6.4.** Find conditions on a point \((x_0, y_0, u_0, v_0)\) such that there exist real-valued functions \( u(x, y) \) and \( v(x, y) \) which are continuously differentiable near \((x_0, y_0)\) and satisfy the simultaneous equations

\[
\begin{align*}
    xu^2 + yv^2 + xy &= 9 \\
    xv^2 + yu^2 - xy &= 7.
\end{align*}
\]
Prove that the solutions satisfy \( u^2 + v^2 = 16/(x + y) \).
11.6.5. Given nonzero numbers \(x_0, y_0, u_0, v_0, s_0, t_0\) which satisfy the simultaneous equations

\[
\begin{align*}
  u^2 + sx + ty &= 0 \\
  v^2 + tx + sy &= 0 \\
  2s^2x + 2t^2y - 1 &= 0 \\
  s^2x - t^2y &= 0,
\end{align*}
\]

prove that there exist functions \(u(x, y), v(x, y), s(x, y), t(x, y)\), and an open ball \(B\) containing \((x_0, y_0)\), such that \(u, v, s, t\) are continuously differentiable and satisfy \((*)\) on \(B\), and such that \(u(x_0, y_0) = u_0, v(x_0, y_0) = v_0, s(x_0, y_0) = s_0,\) and \(t(x_0, y_0) = t_0\).

11.6.6. Let \(E = \{(x, y) : 0 < y < x\}\) and set \(f(x, y) = (x + y, xy)\) for \((x, y) \in E\).

a) Prove that \(f\) is 1–1 from \(E\) onto \(\{(s, t) : s > 2\sqrt{r}, r > 0\}\) and find a formula for \(f^{-1}(s, t)\).

b) Use the Inverse Function Theorem to compute \(D(f^{-1})(f(x, y))\) for \((x, y) \in E\).

c) Use the formula you obtained in part a) to compute \(D(f^{-1})(s, t)\) directly. Check to see that this answer agrees with the one you found in part b).

11.6.7. Suppose that \(V\) is open in \(\mathbb{R}^n\), that \(a \in V\), and that \(F : V \to \mathbb{R}\) is \(C^1\) on \(V\). If \(F(a) = 0 \neq F_{x_j}(a)\) and \(u^{(j)} := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)\) for \(j = 1, 2, \ldots, n\), prove that there exist open sets \(W_j\) containing \((a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n)\), an \(r > 0\), and functions \(g_j(u^{(j)})\), \(C^1\) on \(W_j\), such that \(F(x_1, \ldots, x_{j-1}, g_j(u^{(j)}), x_{j+1}, \ldots, x_n) = 0\) on \(W_j\) and

\[
\frac{\partial g_1}{\partial x_n} \frac{\partial g_2}{\partial x_1} \frac{\partial g_3}{\partial x_2} \cdots \frac{\partial g_n}{\partial x_{n-1}} = (-1)^n
\]
on \(B_r(a)\).

11.6.8. Suppose that \(f : \mathbb{R}^2 \to \mathbb{R}^2\) has continuous first-order partial derivatives in some ball \(B_r(x_0, y_0)\), \(r > 0\). Prove that if \(\Delta f(x_0, y_0) \neq 0\), then

\[
\frac{\partial f_1^{-1}}{\partial x}(f(x_0, y_0)) = \frac{\partial f_2}{\partial y}(f(x_0, y_0)), \quad \frac{\partial f_1^{-1}}{\partial y}(f(x_0, y_0)) = -\frac{\partial f_2}{\partial x}(f(x_0, y_0))
\]

and

\[
\frac{\partial f_2^{-1}}{\partial x}(f(x_0, y_0)) = -\frac{\partial f_1}{\partial y}(f(x_0, y_0)), \quad \frac{\partial f_2^{-1}}{\partial y}(f(x_0, y_0)) = \frac{\partial f_1}{\partial x}(f(x_0, y_0)).
\]

11.6.9. This exercise is used in Section 11.7. Let \(F : \mathbb{R}^3 \to \mathbb{R}\) be continuously differentiable in some open set containing \((a, b, c)\) with \(F(a, b, c) = 0\) and \(\nabla F(a, b, c) \neq 0\).

11.7 OPTIMIZATION

This section contains sections of several definitions.

11.50 Definition

Let \(V\) be a convex open subset of \(\mathbb{R}^n\),

i) \(f(a)\) is defined.

ii) \(f(a)\) is not defined.

iii) \(f(a)\) is defined.

The folklore of real-valued functions in \(\mathbb{R}^n\) is summarized below.

11.51 Theorem

Suppose \(f : \mathbb{R}^n \to \mathbb{R}\) is a real-valued function on an open subset \(V \subset \mathbb{R}^n\), and \(a \in V\).

Proof. The theorem states that if \(f\) has a local extrema at \(a\), then...

11.52 Corollary

In three-dimensional space, if a function has a local extrema, then...

Proof. The corollary follows from the theorem by...

11.53 Example

Consider the function \(f(x, y, z) = x^2 + y^2 - z^2\) on the open set \(V = \{(x, y, z) : x^2 + y^2 > z^2\}\). The function has a local extrema at \((0, 0, 0)\).
a) Prove that the graph of the relation \( F(x, y, z) = 0 \); that is, that the set \( \mathcal{G} := \{(x, y, z): F(x, y, z) = 0\} \) has a tangent plane at \((a, b, c)\).

b) Prove that a normal of the tangent plane to \( \mathcal{G} \) at \((a, b, c)\) is given by \( \nabla F(a, b, c) \).

11.6.10. Suppose that \( f := (u, v): \mathbb{R} \to \mathbb{R}^2 \) is \( C^2 \) and that \((x_0, y_0) = f(t_0)\).

a) Prove that if \( f'(t_0) \neq 0 \), then \( u'(t_0) \) and \( v'(t_0) \) cannot both be zero.

b) If \( f'(t_0) \neq 0 \), show that either there is a \( C^1 \) function \( g \) such that \( g(x_0) = t_0 \) and \( u(g(x)) = x \) for \( x \) near \( x_0 \), or there is a \( C^1 \) function \( h \) such that \( h(y_0) = t_0 \) and \( v(h(y)) = y \) for \( y \) near \( y_0 \).

11.6.11. Let \( \mathcal{H} \) be the hyperboloid of one sheet, given by \( x^2 + y^2 - z^2 = 1 \).

a) Use Exercise 11.6.9 to prove that at every point \((a, b, c) \in \mathcal{H}\), \( \mathcal{H} \) has a tangent plane whose normal is given by \((-a, -b, c)\).

b) Find an equation of each plane tangent to \( \mathcal{H} \) which is perpendicular to the \( xy \)-plane.

c) Find an equation of each plane tangent to \( \mathcal{H} \) which is parallel to the plane \( x + y - z = 1 \).

**11.7 OPTIMIZATION**

This section uses no material from any other enrichment section.

In this section we discuss how to find extreme values of differentiable functions of several variables.

11.50 Definition.

Let \( V \) be open in \( \mathbb{R}^n \), let \( a \in V \), and suppose that \( f: V \to \mathbb{R} \).

i) \( f(a) \) is called a local minimum of \( f \) if and only if there is an \( r > 0 \) such that \( f(a) \leq f(x) \) for all \( x \in B_r(a) \).

ii) \( f(a) \) is called a local maximum of \( f \) if and only if there is an \( r > 0 \) such that \( f(a) \geq f(x) \) for all \( x \in B_r(a) \).

iii) \( f(a) \) is called a local extremum of \( f \) if and only if \( f(a) \) is a local maximum or a local minimum of \( f \).

The following result shows that, as in the one-dimensional case, extrema of real-valued differentiable functions occur among points where the "derivative" is zero.

11.51 Remark. If the first-order partial derivatives of \( f \) exist at \( a \), and \( f(a) \) is a local extremum of \( f \), then \( \nabla f(a) = 0 \).

Proof. The one-dimensional function \( g(t) = f(a_1, \ldots, a_{j-1}, t, a_{j+1}, \ldots, a_n) \) has a local extremum at \( t = a_j \) for each \( j = 1, \ldots, n \). Hence, by the one-dimensional theory,