EXERCISES

12.1.1. a) For $m = 1, 2, 3$, let $\mathcal{G}_m$ be the grid on $[0, 1] \times [0, 1]$ generated by

$$\mathcal{P}_j(\mathcal{G}_m) = \{k/2^m : k = 0, 1, \ldots, 2^m\},$$

where $j = 1, 2$. For each of the following sets, compute $V(E; \mathcal{G}_m)$.

α) $E = \{(x, y) \in [0, 1] \times [0, 1] : x = 0 \text{ or } y = 0\}$.

β) $E = \{(x, y) \in [0, 1] \times [0, 1] : y \leq x\}$.

γ) $E = \{(x, y) \in [0, 1] \times [0, 1] : (2x - 1)^2 + (2y - 1)^2 \leq 1\}$.

b) For each E in part a), compute $v(E; \mathcal{G}_m)$.

12.1.2. a) Prove that every finite subset of $\mathbf{R}^n$ is a Jordan region of volume zero.

b) Show that, even in $\mathbf{R}^2$, part a) is not true if *finite* is replaced by *countable*.

c) By an interval in $\mathbf{R}^2$ we mean a set of the form

$$\{(x, c) : a \leq x \leq b\} \quad \text{or} \quad \{(c, y) : a \leq y \leq b\}$$

for some $a, b, c \in \mathbf{R}$. Prove that every interval in $\mathbf{R}^2$ is a Jordan region.

12.1.3. Prove that every grid is a nonoverlapping collection of Jordan regions.

12.1.4. a) Prove that the boundary of an open ball $B_r(\mathbf{a})$ is given by

$$\partial B_r(\mathbf{a}) = \{\mathbf{x} : \|\mathbf{x} - \mathbf{a}\| = r\}.$$

b) Prove that $B_r(\mathbf{a})$ is a Jordan region for all $\mathbf{a} \in \mathbf{R}^n$ and all $r \geq 0$.

12.1.5. Let E be a Jordan region in $\mathbf{R}^n$.

a) Prove that E^o and $\overline{E}$ are Jordan regions.

$\boxed{\text{b)}}$ **This exercise is used in Section 12.2.** Prove that $\text{Vol}(E^o) = \text{Vol}(\overline{E}) = \text{Vol}(E)$.

c) Prove that $\text{Vol}(E) > 0$ if and only if $E^o \neq \emptyset$.

d) Let $f : [a, b] \to \mathbf{R}$ be continuous on $[a, b]$. Prove that the graph of $y = f(x)$, $x \in [a, b]$, is a Jordan region in $\mathbf{R}^2$.

e) Does part d) hold if *continuous* is replaced by *integrable*? How about *bounded*?

$\boxed{\textbf{12.1.6}}$**.** **This exercise is used in Section *12.5.** Suppose that E_1, E_2 are Jordan regions in $\mathbf{R}^n$.

a) Prove that if $E_1 \subseteq E_2$, then

$$\text{Vol}(E_1) \leq \text{Vol}(E_2).$$

b) Prove that $E_1 \cap E_2$ and $E_1 \setminus E_2$ are Jordan regions.

c) Prove that if E_1, E_2 are nonoverlapping, then

$$\text{Vol}(E_1 \cup E_2) = \text{Vol}(E_1) + \text{Vol}(E_2).$$

d) If $E_2 \subseteq E_1$, prove that

$$\operatorname{Vol}(E_1 \setminus E_2) = \operatorname{Vol}(E_1) - \operatorname{Vol}(E_2).$$

e) Prove that

$$\operatorname{Vol}(E_1 \cup E_2) = \operatorname{Vol}(E_1) + \operatorname{Vol}(E_2) - \operatorname{Vol}(E_1 \cap E_2).$$

$\boxed{\textbf{12.1.7}}$. **This exercise is used in Section *12.6.** Let $E \subset \mathbf{R}^n$. The *translation* of E by an $\mathbf{x} \in \mathbf{R}^n$ is the set $\mathbf{x} + E = \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} = \mathbf{x} + \mathbf{z} \text{ for some } \mathbf{z} \in E\}$, and the *dilation* of E by a scalar $\alpha > 0$ is the set $\alpha E = \{\mathbf{y} \in \mathbf{R}^n : \mathbf{y} = \alpha\mathbf{z} \text{ for some } \mathbf{z} \in E\}$.

a) Prove that E is a Jordan region if and only if $\mathbf{x}+E$ is a Jordan region, in which case $\operatorname{Vol}(\mathbf{x} + E) = \operatorname{Vol}(E)$.
b) Prove that E is a Jordan region if and only if αE is a Jordan region, in which case $\operatorname{Vol}(\alpha E) = \alpha^n \operatorname{Vol}(E)$.

12.1.8. A set $E \subset \mathbf{R}^n$ is said to be of *measure zero* if and only if given $\varepsilon > 0$ there is a sequence of rectangles $R_1, R_2, \dots$ which covers E such that $\sum_{k=1}^{\infty} |R_k| < \varepsilon$.

a) Prove that if $E \subset \mathbf{R}^n$ is of volume zero, then E is of measure zero.
b) Prove that if $E \subset \mathbf{R}^n$ is at most countable, then E is of measure zero.
c) Prove that there is a set $E \subset \mathbf{R}^2$ of measure zero which does not have zero area and, in fact, is not even a Jordan region.

***12.1.9.** Show that if $E \subset \mathbf{R}^n$ is bounded and has only finitely many cluster points, then E is a Jordan region.

12.2 RIEMANN INTEGRATION ON JORDAN REGIONS

By analogy with the one-variable case, the integral of a nonnegative function f over a Jordan region E should be the volume of the set $\{(\mathbf{x}, t) : \mathbf{x} \in E, 0 \le t \le f(\mathbf{x})\}$. We should be able to approximate this volume by using $(n+1)$-dimensional rectangles whose heights approximate $t = f(\mathbf{x})$ and whose bases belong to some grid on E (see Figure 12.5). This leads us to the following definition (compare with Definition 5.13).

12.15 Definition.

Let E be a Jordan region in $\mathbf{R}^n$, let $f : E \to \mathbf{R}$ be a bounded function, let R be an n-dimensional rectangle such that $E \subseteq R$, and let $\mathcal{G} = \{R_1, \dots, R_p\}$ be a grid on R. Extend f to $\mathbf{R}^n$ by setting $f(\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbf{R}^n \setminus E$.

i) The *upper sum* of f on E with respect to $\mathcal{G}$ is

$$U(f, \mathcal{G}) := \sum_{R_j \cap E \neq \emptyset} M_j |R_j|,$$

where $M_j = \sup_{\mathbf{x} \in R_j} f(\mathbf{x})$.

ii) *Suppose that V is an open set in $\mathbf{R}^n$ such that $\overline{E} \subset V$, and that $\boldsymbol{\phi} : V \to \mathbf{R}^n$ is 1–1 and $\boldsymbol{\phi}^{-1}$ is $\mathcal{C}^1$ on $\boldsymbol{\phi}(V)$ with $\Delta_{\boldsymbol{\phi}^{-1}} \neq 0$. If f is integrable on $\boldsymbol{\phi}(E)$, then $f \circ \boldsymbol{\phi}$ is integrable on E.*

Proof. i) This part can be verified by modifying the proof of Theorem 9.49 (see Spivak [12], p. 53).

ii) By part i) and Theorem 12.10, it suffices to show that the set of points of discontinuity of $f \circ \boldsymbol{\phi}$ on E is a set of measure zero. Let $\varepsilon > 0$. Since f is integrable on $\boldsymbol{\phi}(E)$, its set of points of discontinuity, D, can be covered by cubes Q_k such that $\sum_{k=1}^{\infty} |Q_k| < \varepsilon$. Set $\boldsymbol{\psi} = \boldsymbol{\phi}^{-1}$ and apply (2), with $\boldsymbol{\psi}$ in place of $\boldsymbol{\phi}$, to choose an absolute constant C and cubes Q_k^{ψ} such that $\psi(Q_k) \subseteq Q_k^{\psi}$ and $|Q_k^{\psi}| \leq C|Q_k|$. Then $\{Q_k^{\psi}\}$ covers $\psi(D) = \boldsymbol{\phi}^{-1}(D)$ and

$$\sum_{k=1}^{\infty} |Q_k^{\psi}| \leq C \sum_{k=1}^{\infty} |Q_k| < C\varepsilon.$$

Hence, $\boldsymbol{\phi}^{-1}(D) := \psi(D)$ is a set of measure zero. But since D is the set of points of discontinuity of f on $\boldsymbol{\phi}(E)$, $\boldsymbol{\phi}^{-1}(D)$ is the set of points of discontinuity of $f \circ \boldsymbol{\phi}$ on E. Hence $f \circ \boldsymbol{\phi}$ is Riemann integrable by part i). ∎

EXERCISES

12.2.1. Using Exercise 1.4.4a, compute the upper and lower sums $U(f, \mathcal{G}_m)$, $L(f, \mathcal{G}_m)$ for $m \in \mathbf{N}$, where $f(x, y) = xy$ and $\mathcal{G}_m$ is determined by

$$\mathcal{P}_j(\mathcal{G}_m) = \{k/2^m : k = 0, 1, \dots, 2^m\}$$

for $j = 1, 2$. Prove that

$$\lim_{m \to \infty} U(f, \mathcal{G}_m) - L(f, \mathcal{G}_m) = 0.$$

12.2.2. Let E be a Jordan region in $\mathbf{R}^n$ with $E \subseteq [0, 1] \times \cdots \times [0, 1]$. If f, g are integrable on E with

$$\int_E f\, dV = 1 \quad \text{and} \quad \int_E g\, dV = -1,$$

and if $g(\mathbf{x}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in E$, prove that for each $j \in \{1, 2, \dots, n\}$ there is a $0 \leq t_j \leq 2$ such that

$$\int_E x_j^2 (f(\mathbf{x}) - g(\mathbf{x}))\, d\mathbf{x} = t_j.$$

$\boxed{\textbf{12.2.3}}$. **This exercise is used in Sections 12.4, 13.5, and 13.6.** Let E be an open Jordan region in $\mathbf{R}^n$ and $\mathbf{x}_0 \in E$. If $f : E \to \mathbf{R}$ is integrable on E and continuous at $\mathbf{x}_0$, prove that

$$\lim_{r\to 0+}\frac{1}{\mathrm{Vol}(B_r(\mathbf{x}_0))}\int_{B_r(\mathbf{x}_0)} f\,dV = f(\mathbf{x}_0).$$

12.2.4. a) Suppose that E is a Jordan region in $\mathbf{R}^n$ and that $f_k : E \to \mathbf{R}$ are integrable on E for $k \in \mathbf{N}$. If $f_k \to f$ uniformly on E as $k \to \infty$, prove that f is integrable on E and

$$\lim_{k\to\infty}\int_E f_k(\mathbf{x})\,d\mathbf{x} = \int_E f(\mathbf{x})\,d\mathbf{x}.$$

b) Prove that

$$\lim_{k\to\infty}\iint_E \cos(x/k)e^{y/k}\,dA$$

exists, and find its value for any Jordan region E in $\mathbf{R}^2$.

12.2.5. If $E_0 \subset E$ are Jordan regions in $\mathbf{R}^n$ and $f : E \to \mathbf{R}$ is integrable on E, prove that f is integrable on E_0.

12.2.6. Let H be a closed, connected, nonempty Jordan region and suppose that $f : H \to \mathbf{R}$ is continuous. If $g : H \to \mathbf{R}$ is integrable and nonnegative on H, prove that there is an $\mathbf{x}_0 \in H$ such that

$$f(\mathbf{x}_0)\int_H g(\mathbf{x})\,d\mathbf{x} = \int_H f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x}.$$

12.2.7. Suppose that $Q := \{(x, y) \in \mathbf{R} : x > 0 \text{ and } y > 0\}$ and that f is a continuous function on $\mathbf{R}^2$ whose first-order partial derivative satisfies $|f_x| \le 1$. If

$$F(x, y) := \frac{1}{x^3}\iint_{B_x(0,0)} (f(u, y) - f(v, y))\,d(u, v)$$

for $(x, y) \in Q$, prove that F is bounded on Q.
[Hint: You may use polar coordinates to change variables in F.]

12.2.8. Suppose that E is a Jordan region in $\mathbf{R}^n$ and that $f, g : E \to \mathbf{R}$ is integrable on E.

a) Modifying the proof of Corollary 5.23, prove that fg is integrable on E.

b) Prove that $f \vee g$ and $f \wedge g$ are integrable on E (see Exercise 3.1.8).

12.2.9. Suppose that V is open in $\mathbf{R}^n$ and that $f : V \to \mathbf{R}$ is continuous. Prove that if

$$\int_E f\,dV = 0$$

for all nonempty Jordan regions $E \subset V$, then $f = 0$ on V.

12.2.10. Suppose that E is a Jordan region and that $f : E \to \mathbf{R}$ is integrable.

a) If $f(E) \subseteq H$, for some compact set H, and $\phi : H \to \mathbf{R}$ is continuous, prove that $\phi \circ f$ is integrable on E.

*b) Show that part a) is false if ϕ has even one point of discontinuity.

12.2.11. Prove the following special case of Theorem 12.29i. Suppose that E and E_0 are a Jordan regions in $\mathbf{R}^n$, and that $f : E \to \mathbf{R}$ is bounded. If f is continuous on $E \setminus E_0$, then f is integrable on E.

12.3 ITERATED INTEGRALS

If $f(x_1, \dots, x_k, \dots, x_j, \dots, x_n)$ is defined for $x_k \in [c, d]$ and $x_j \in [a, b]$, $j \neq k$, then we shall call

$$\int_c^d \int_a^b f(x_1, \dots, x_n)\, dx_j\, dx_k := \int_c^d \left(\int_a^b f(x_1, \dots, x_n)\, dx_j \right) dx_k$$

an *iterated integral*, when the integrals on the right side exist. In a similar way, we define higher-order iterated integrals.

In the preceding section we defined the Riemann integral of a multivariable function but developed no practical way to evaluate it. In this section we show that, for a large collection of Jordan regions E, integrals over E can be evaluated using iterated integrals.

For simplicity, we begin with the two-dimensional case. Recall that if $\phi : [a, b] \to \mathbf{R}$ is bounded, then the upper Riemann integral, $(U) \int_a^b \phi(x)\, dx$, and the lower Riemann integral, $(L) \int_a^b \phi(x)\, dx$, both exist and are finite.

12.30 *Lemma.*
Let $R = [a, b] \times [c, d]$ be a two-dimensional rectangle and suppose that $f : R \to \mathbf{R}$ is bounded. If $f(x, \cdot)$ is integrable on $[c, d]$ for each $x \in [a, b]$, then

$$\begin{aligned} (L) \iint_R f\, dA &\le (L) \int_a^b \left(\int_c^d f(x, y) dy \right) dx \\ &\le (U) \int_a^b \left(\int_c^d f(x, y) dy \right) dx \le (U) \iint_R f\, dA. \end{aligned} \tag{22}$$

Proof. Let $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, where $\{x_0, \dots, x_k\}$ is a partition of $[a, b]$ and $\{y_0, \dots, y_\ell\}$ is a partition of $[c, d]$. Then $\mathcal{G} = \{R_{ij} : i = 1, 2, \dots, k, j = 1, 2 \dots, \ell\}$ is a grid on R.

Let $\varepsilon > 0$, choose $\mathcal{G}$ so that

$$U(f, \mathcal{G}) - \varepsilon < (U) \iint_R f\, dA, \tag{23}$$

and set

$$M_{ij} = \sup_{(x,y) \in R_{ij}} f(x, y). \tag{24}$$

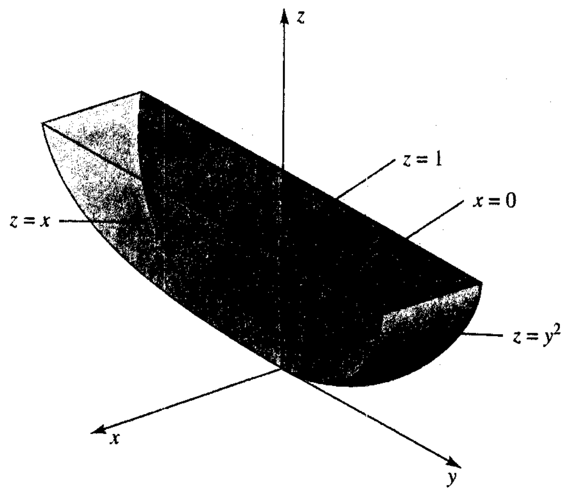

FIGURE 12.11

12.42 *EXAMPLE.*

Find the integral of $f(x, y, z) = x - z$ over the region bounded by $z = y^2$, $z = 1$, $z = x$, and $x = 0$.

Solution. The region E is a union of two regions of type I (see Figure 12.11, where the "back" of E is that portion of the plane $x = 0$ which is bounded by the parabola $z = y^2$, $x = 0$ here represented by a dashed line). Therefore, we must use two integrals if we integrate dz first: the integral where z varies between y^2 and 1, and the integral where z varies from x to 1. It looks complicated to set up. The solution is simpler if we integrate dx first. Indeed, E is a single region of type III since

$$E = \{(x, y, z) : -1 \le y \le 1,\ y^2 \le z \le 1,\ 0 \le x \le z\}.$$

Thus,

$$\begin{aligned}\iiint_E f\,dV &= \int_{-1}^{1}\int_{y^2}^{1}\int_{0}^{z}(x - z)\,dx\,dz\,dy \\ &= -\frac{1}{2}\int_{-1}^{1}\int_{y^2}^{1} z^2\,dz\,dy = \frac{1}{6}\int_{-1}^{1}(y^6 - 1)\,dy = -\frac{2}{7}. \qquad \blacksquare\end{aligned}$$

EXERCISES

12.3.1. Evaluate each of the following iterated integrals.

a) $\int_0^1 \int_0^1 (x + y)\,dx\,dy$

b) $\int_0^3 \int_0^1 \sqrt{xy+x}\, dx\, dy$ 12.3.7.

c) $\int_0^{\pi} \int_0^{\pi} y\cos(xy)\, dy\, dx$

12.3.2. Evaluate each of the following iterated integrals. Write each as an integral over a region E, and sketch E in each case.

a) $\int_0^1 \int_x^{x^2+1} (x+1)\, dy\, dx$

b) $\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy$

c) $\int_0^1 \int_{\sqrt{y}}^1 \int_0^{x^2+y^2} dz\, dx\, dy$

d) $\int_0^1 \int_{\sqrt{y}}^1 \int_{x^3}^1 \sqrt{x^3+z}\, dz\, dx\, dy$

12.3.3. For each of the following, evaluate $\int_E f\, dV$. 12.3.8.

a) $f(x,y) = (1+x^2)^{-1}$ and E is bounded by $x=1,\ y=0$, and $y=x^3$.

b) $f(x,y) = x+y$ and E is the triangle with vertices $(0,0)$, $(0,1)$, and $(2,0)$.

c) $f(x,y) = x^2e^{xy}$ and E is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,1)$.

d) $f(x,y,z) = x$ and E is the set of points (x,y,z) such that $0 \le z \le 1-x^2,\ 0 \le y \le x^2+z^2$, and $x \ge 0$. 12.3.9.

12.3.4. Compute the volume of each of the following regions.

a) E is bounded by the surfaces $x+y+z=3,\ z=0$, and $x^2+y^2=1$.

b) E lies under the plane $z=x+y$ and over the region in the xy-plane bounded by the curves $x=\sqrt{y/2},\ x=2\sqrt{y},\ x+y=3$.

c) E is bounded by $z=y^2,\ x=y^2+z^2,\ x=0,\ z=1$.

d) E is bounded by $y=x^3,\ x=z^2,\ z=x^2$, and $y=0$.

12.3.5. a) Verify that the hypotheses of Fubini's Theorem hold when f is continuous on R.

b) Modify the proof of Remark 12.33 to show that Fubini's Theorem might not hold for a nonintegrable f, even if $f(x,y)$ is continuous in each variable separately; that is, if $f(x,\cdot)$ is continuous for each $x \in [a,b]$ and $f(\cdot,y)$ is continuous for each $y \in [c,d]$.

12.3.6. a) Suppose that f_k is integrable on $[a_k,b_k]$ for $k=1,\dots,n$, and set $R=[a_1,b_1]\times\cdots\times[a_n,b_n]$. Prove that

$$\int_R f_1(x_1)\dots f_n(x_n)\, d(x_1,\dots,x_n) = \left(\int_{a_1}^{b_1} f_1(x_1)\, dx_1\right)\dots\left(\int_{a_n}^{b_n} f_n(x_n)\, dx_n\right).$$

b) If $Q=[0,1]^n$ and $\mathbf{y} := (1,1,\dots,1)$, prove that

$$\int_Q e^{-\mathbf{x}\cdot\mathbf{y}}\, d\mathbf{x} = \left(\frac{e-1}{e}\right)^n.$$

12.3.7. The greatest integer in a real number x is the integer $[x] := n$ which satisfies $n \le x < n+1$. An interval $[a,b]$ is called **Z**-asymmetric if $b+a \ne [b]+[a]+1$.

a) Suppose that R is a two-dimensional **Z**-asymmetric rectangle (i.e., that both of its sides are **Z**-asymmetric). If $\psi(x,y) := (x-[x]-1/2)(y-[y]-1/2)$, prove that $\iint_R \psi\, dA = 0$ if and only if at least one side of R has integer length.

b) Suppose that R is tiled by rectangles $R_1 \dots, R_N$ (i.e., that the R_j's are **Z**-asymmetric, nonoverlapping, and that $R = \cup_{j=1}^N R_j$). Prove that if each R_j has at least one side of integer length and R is **Z**- asymmetric, then R has at least one side of integer length.

12.3.8. Let E be a nonempty Jordan region in $\mathbf{R}^2$ and $f : E \to [0,\infty)$ be integrable on E. Prove that the volume of $\Omega = \{(x,y,z) : (x,y) \in E, 0 \le z \le f(x,y)\}$ (as given by Definition 12.5) satisfies

$$\operatorname{Vol}(\Omega) = \iint_E f\, dA.$$

12.3.9. Let $R = [a,b] \times [c,d]$ be a two-dimensional rectangle and $f : R \to \mathbf{R}$ be bounded.

a) Prove that

$$\begin{aligned}(L)\iint_R f\,dA &\le (L)\int_a^b \left((X)\int_c^d f(x,y)\,dy\right) dx\\ &\le (U)\int_a^b \left((X)\int_c^d f(x,y)\,dy\right) dx\\ &\le (U)\iint_R f\,dA\end{aligned}$$

for $X = U$ or $X = L$.

b) Prove that if f is integrable on R, then

$$\begin{aligned}\iint_R f\,dA &= \int_a^b \left((L)\int_c^d f(x,y)\,dy\right) dx\\ &= \int_a^b \left((U)\int_c^d f(x,y)\,dy\right) dx.\end{aligned}$$

c) Compute the two iterated integrals in part b) for

$$f(x,y) = \begin{cases} 1 & y \in \mathbf{Q} \\ x & y \notin \mathbf{Q}\end{cases}$$

and $R = [0,1] \times [0,1]$. Prove that f is not integrable on R.

***12.3.10.** [FUBINI'S THEOREM FOR IMPROPER INTEGRALS]. If $a < b$ are extended real numbers, $c < d$ are finite real numbers, $f : (a, b) \times [c, d] \to \mathbf{R}$ is continuous, and

$$F(y) = \int_a^b f(x, y)\, dx$$

converges uniformly on $[c, d]$, prove that

$$\int_c^d f(x, y)\, dy$$

is improperly integrable on (a, b) and

$$\int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

12.4 CHANGE OF VARIABLES

Recall (Theorem 5.34) that if $\phi : [a, b] \to \mathbf{R}$ is continuously differentiable and $\phi' \neq 0$ on $[a, b]$, then

$$\int_{\phi([a,b])} f(t)\, dt = \int_{[a,b]} f(\phi(x))\, |\phi'(x)|\, dx$$

for all f integrable on $\phi([a, b])$. We shall generalize this result to functions of several variables; namely, we shall identify conditions under which

$$\int_{\boldsymbol{\phi}(E)} f(\mathbf{u})\, d\mathbf{u} = \int_E f(\boldsymbol{\phi}(\mathbf{x}))|\Delta_{\boldsymbol{\phi}}(\mathbf{x})|\, d\mathbf{x} \tag{30}$$

holds. (At this point you may wish to read the discussion following the proof of Theorem 12.46 below to see that $\Delta_{\boldsymbol{\phi}}$ takes on a familiar form when $\boldsymbol{\phi}$ is the change from polar to rectangular coordinates.)

It takes six or seven hypotheses to establish (30). These hypotheses fall into two categories:

1) *Hypotheses made so the change of variables is possible.* Since the one-dimensional result required ϕ to be continuously differentiable and $\phi' \neq 0$ (which together imply that ϕ is 1–1), we expect the corresponding hypotheses for (30) to be as follows: $\boldsymbol{\phi}$ is 1–1, continuously differentiable, and $\Delta_{\boldsymbol{\phi}} \neq 0$.
2) *Hypotheses made so the integrals in (30) exist.* There are four of these: E is a Jordan region, $\boldsymbol{\phi}(E)$ is a Jordan region, f is integrable on $\boldsymbol{\phi}(E)$, and $f \circ \boldsymbol{\phi}|\Delta_{\boldsymbol{\phi}}|$ is integrable on E. In practice, only the first and third of the hypotheses in category 2) need be verified. Indeed, if $\boldsymbol{\phi}$ satisfies all hypotheses in category 1) and E is Jordan, then $\boldsymbol{\phi}(E)$ is Jordan by Theorem 12.10, and, when f is integrable on $\boldsymbol{\phi}(E)$, $f \circ \boldsymbol{\phi}|\Delta_{\boldsymbol{\phi}}|$ is integrable on E (see Theorem 12.29ii and Exercise 12.2.8a). Moreover, the remaining hypotheses in category 2) can usually be verified by inspection. The reason for this is twofold. E is frequently projectable, hence a Jordan region, and most functions are

where E is the region bounded by $y = 2x - 1$, $y = 2x + 3$, $y = -x$, and $y = -x + 1$.

Solution. Let $\phi(x, y) = (2x - y, x + y)$ and observe that the integral in question looks like the right side of (38) except the Jacobian is missing. By Cramer's Rule, for each fixed $u, v \in \mathbf{R}$, the system $u = 2x - y$, $v = x + y$ has a unique solution in x, y. Hence, ϕ is 1–1 on $\mathbf{R}^2$. It is obviously continuously differentiable, and its Jacobian,

$$\Delta_\phi(x, y) = \frac{\partial(u, v)}{\partial(x, y)} = \det\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} = 3,$$

is a nonzero constant. Hence, we can make adjustments to the integral in question so that it is precisely the right side of (38):

$$\iint_E \sin(x + y)\cos(2x - y)\, dA = \frac{1}{3}\iint_E f \circ \phi(x, y)\Delta_\phi(x, y)\, d(x, y),$$

where $f(u, v) = \cos u \sin v$. It remains to compute the left side of (38) (i.e., to find what happens to E under ϕ).

Notice that $y = 2x - 1$ implies $u = 1$, $y = 2x + 3$ implies $u = -3$, $y = -x$ implies $v = 0$, and $y = -x + 1$ implies $v = 1$. Thus $\phi(E) = [-3, 1] \times [0, 1]$. Applying Theorem 12.46 and the preliminary step taken above, we find

$$\begin{aligned}\iint_E \sin(x + y)\cos(2x - y)\, dA &= \frac{1}{3}\int_0^1 \int_{-3}^1 \sin v \cos u\, du\, dv \\ &= \frac{1}{3}(\sin(1) + \sin(3))(1 - \cos(1)). \qquad \blacksquare\end{aligned}$$

EXERCISES

12.4.1. Evaluate each of the following integrals.

a) $$\int_0^2 \int_0^{\sqrt{4-x^2}} \sin(x^2 + y^2)\, dy\, dx$$

b) $$\int_0^1 \int_0^x \sqrt[3]{(2y - y^2)^2}\, dy\, dx$$

c) $$\int_a^b \int_0^x \sqrt{x^2 + y^2}\, dy\, dx, \qquad 0 \le a < b$$

12.4.2. For each of the following, find $\iint_E f\, dA$.

a) $f(x, y) = \cos(3x^2 + y^2)$ and E is the set of points satisfying $x^2 + y^2/3 \le 1$.

b) $f(x, y) = y\sqrt{x - 2y}$ and E is bounded by the triangle with vertices $(0, 0)$, $(4, 0)$, and $(4, 2)$.

12.4.3. For each of the following, find $\iiint_E f\, dV$.

a) $f(x, y, z) = z^2$ and E is the set of points satisfying $x^2 + y^2 + z^2 \le 6$ and $z \ge x^2 + y^2$.

b) $f(x, y, z) = e^z$ and E is the set of points satisfying $x^2 + y^2 + z^2 \le 9$, $x^2 + y^2 \le 1$, and $z \ge 0$.

c) $f(x, y, z) = (x - y)z$ and E is the set of points satisfying $x^2 + y^2 + z^2 \le 4$, $z \ge \sqrt{x^2 + y^2}$, and $x \ge 0$.

12.4.4. a) Prove that the volume bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $4\pi abc/3$.

b) Let a, b, c, d be positive numbers and $r^2 < d^2/(b^2 + c^2)$. Find the volume of the region bounded by $y^2 + z^2 = r^2$, $x = 0$, and $ax + by + cz = d$.

c) Show that for any $a \ge 0$, the volume of the region bounded by the cylinders $x^2 + z^2 = a^2$ and $y^2 + z^2 = a^2$ is $16a^3/3$.

12.4.5. a) Compute $\iint_E \sqrt{x - y}\sqrt{x + 2y}\, dA$, where E is the parallelogram with vertices $(0, 0)$, $(2/3, -1/3)$, $(1, 0)$, $(1/3, 1/3)$.

b) Compute $\iint_E \sqrt[3]{2x^2 - 5xy - 3y^2}\, dA$, where E is the parallelogram bounded by the lines $y = x/3$, $y = (x-1)/3$, $y = -2x$, $y = 1-2x$.

c) Find

$$\iint_E e^{(y-x)/(y+x)}\, dA,$$

where E is the trapezoid with vertices $(1, 1)$, $(2, 2)$, $(2, 0)$, $(4, 0)$.

d) Given $\int_0^1 (1 - x) f(x)\, dx = 5$, find

$$\int_0^1 \int_0^x f(x - y)\, dy\, dx.$$

12.4.6. Suppose that V is nonempty and open in $\mathbf{R}^n$ and that $\mathbf{f} : V \to \mathbf{R}^n$ is continuously differentiable with $\Delta_{\mathbf{f}} \ne 0$ on V. Prove that

$$\lim_{r \to 0+} \frac{\operatorname{Vol}(\mathbf{f}(B_r(\mathbf{x}_0)))}{\operatorname{Vol}(B_r(\mathbf{x}_0))} = |\Delta_{\mathbf{f}}(\mathbf{x}_0)|$$

for every $\mathbf{x}_0 \in V$.

12.4.7. Show that Vol is *rotation invariant* in $\mathbf{R}^2$; that is, if ϕ is a rotation on $\mathbf{R}^2$ (see Exercise 8.2.9) and E is a Jordan region in $\mathbf{R}^2$, then

$$\operatorname{Vol}(\phi(E)) = \operatorname{Vol}(E).$$

12.4.8. a) Compute the Jacobian of the change of variables from spherical coordinates to rectangular coordinates.

b) Assuming that Vol is translation and rotation invariant (see Exercises 12.1.7 and 12.4.7), verify the following classical formulas: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$, and the volume of a right circular cone of altitude h and radius r is $\pi r^2 h/3$.

12.4.9. Let $\mathbf{v}_j = (v_{j1}, \dots, v_{jn}) \in \mathbf{R}^n$, $j = 1, \dots, n$, be fixed. The *parallelepiped* determined by the vectors $\mathbf{v}_j$ is the set

$$\mathcal{P}(\mathbf{v}_1, \dots, \mathbf{v}_n) := \{t_1\mathbf{v}_1 + \cdots + t_n\mathbf{v}_n : t_j \in [0, 1]\},$$

and the determinant of the $\mathbf{v}_j$'s is the number

$$\det(\mathbf{v}_1, \dots, \mathbf{v}_n) := \det\left[v_{jk}\right]_{n \times n}.$$

Prove that

$$\mathrm{Vol}(\mathcal{P}(\mathbf{v}_1, \dots, \mathbf{v}_n)) = |\det(\mathbf{v}_1, \dots, \mathbf{v}_n)|.$$

Check this formula for $n = 2$ and $n = 3$ to see that it agrees with the classical formulas for the area of a parallelogram and the volume of a parallelepiped.

12.4.10. **This exercise is used in Section *12.6.**

a) Prove that the improper integral $\int_0^\infty e^{-x^2}\,dx$ converges to a finite real number.

b) Prove that if I is the value of the integral in part a), then

$$I^2 = \lim_{N\to\infty} \int_0^{\pi/2} \int_0^N e^{-r^2} r\,dr\,d\theta.$$

c) Show that

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

d) Let Q_k represent the n-dimensional cube $[-k, k] \times \cdots \times [-k, k]$. Find

$$\lim_{k\to\infty} \int_{Q_k} e^{-\|\mathbf{x}\|^2}\,d\mathbf{x}.$$

12.4.11. Let $H \subset V \subset \mathbf{R}^n$, with H convex and V open, and suppose that $\boldsymbol{\phi} : V \to \mathbf{R}^n$ is $\mathcal{C}^1$.

a) Show that if E is a closed subset of H^o and

$$\boldsymbol{\epsilon}_{\mathbf{h}}(\mathbf{x}) := \boldsymbol{\phi}(\mathbf{x}+\mathbf{h}) - \boldsymbol{\phi}(\mathbf{x}) - D\boldsymbol{\phi}(\mathbf{x})(\mathbf{h}), \quad \text{for } \mathbf{x} \in V \text{ and } \mathbf{h} \text{ small},$$

then $\boldsymbol{\epsilon}_{\mathbf{h}}(\mathbf{x})/\|\mathbf{h}\| \to 0$ uniformly on E, as $\mathbf{h} \to \mathbf{0}$.

b) Show that if R is a closed rectangle in H^o and $S := (D\boldsymbol{\phi}(\mathbf{x}))^{-1}$ exists for some $\mathbf{x} \in R$, then given $\varepsilon > 0$ there are constants $\delta > 0$ and $M > 0$ and a function $T(\mathbf{x}, \mathbf{y})$ such that

$$S \circ \boldsymbol{\phi}(\mathbf{x}) - S \circ \boldsymbol{\phi}(\mathbf{y}) = \mathbf{x} - \mathbf{y} + T(\mathbf{x}, \mathbf{y})$$

for $\mathbf{x}, \mathbf{y} \in R$, and $\|T(\mathbf{x}, \mathbf{y})\| \le M\varepsilon$ when $\|\mathbf{x} - \mathbf{y}\| < \delta$.

c) Use parts a) and b) to prove that if $\Delta_{\boldsymbol{\phi}}$ is nonzero on V, $\mathbf{x} \in H^o$, and ε is sufficiently small, then there exist numbers $C_\varepsilon > 0$, which depend only on H, $\boldsymbol{\phi}$, n, and ε, and a $\delta > 0$ such that $C_\varepsilon \to 1$ as $\varepsilon \to 0$ and $\mathrm{Vol}(S \circ \boldsymbol{\phi}(Q)) \le C_\varepsilon |Q|$ for all cubes $Q \subset H$ which contain $\mathbf{x}$ and satisfy $\mathrm{Vol}(Q) < \delta$.

d) Use part c) and Exercise 12.4.9 to prove that if $\Delta_{\boldsymbol{\phi}}$ is nonzero on V and $\mathbf{x} \in H^o$, then given any sequence of cubes Q_j which satisfy $\mathbf{x} \in Q_j$ and $\mathrm{Vol}(Q_j) \to 0$ as $j \to \infty$, it is also the case that $\mathrm{Vol}(\boldsymbol{\phi}(Q_j))/|Q_j| \to |\Delta_{\boldsymbol{\phi}}(\mathbf{x})|$ as $j \to \infty$.

*12.5 PARTITIONS OF UNITY

This section uses results from Section 9.5.

In this section we show that a smooth function can be broken into a sum of smooth functions, each of which is zero except on a small set, and use this to prove a global change-of-variables formula when the Jacobian is nonzero off a set of volume zero. This same technique can be used to prove the Fundamental Theorem of Calculus on manifolds (see [12], for example).

12.52 Definition.

Let $f : \mathbf{R}^n \to \mathbf{R}$.

i) The *support* of f is the closure of the set of points at which f is nonzero; that is,

$$\operatorname{spt} f := \overline{\{\mathbf{x} \in \mathbf{R}^n : f(\mathbf{x}) \neq 0\}}.$$

ii) A function f is said to have *compact support* if and only if spt f is a compact set.

12.53 *EXAMPLE.*

If

$$f(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q}, \end{cases}$$

then $\operatorname{spt} f = \mathbf{R}$.