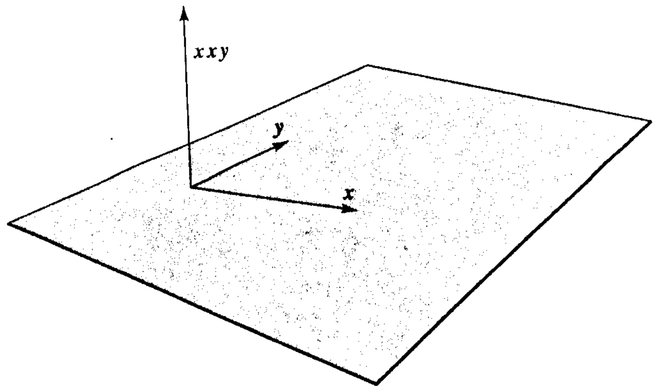

FIGURE 8.3

By (2), there is a close connection between dot products and cosines. The following result shows that there is a similar connection between cross products and sines.

8.10 Remark. *Let* $\mathbf{x}, \mathbf{y}$ *be nonzero vectors in* $\mathbf{R}^3$ *and* θ *be the angle between* $\mathbf{x}$ *and* $\mathbf{y}$. *Then*

$$\|\mathbf{x} \times \mathbf{y}\| = \|\mathbf{x}\| \, \|\mathbf{y}\| \, \sin\theta.$$

Proof. By Theorem 8.9vi and (2),

$$\begin{aligned}\|\mathbf{x} \times \mathbf{y}\|^2 &= (\|\mathbf{x}\| \, \|\mathbf{y}\|)^2 - (\|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta)^2 \\ &= (\|\mathbf{x}\| \, \|\mathbf{y}\|)^2 (1 - \cos^2\theta) = (\|\mathbf{x}\| \, \|\mathbf{y}\|)^2 \sin^2\theta. \end{aligned}$$ ■

This observation can be used to establish a connection between cross products and area or volume (see Exercise 8.2.7).

EXERCISES

8.1.1. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{R}^n$.

a) If $\|\mathbf{x} - \mathbf{z}\| < 2$ and $\|\mathbf{y} - \mathbf{z}\| < 3$, prove that $\|\mathbf{x} - \mathbf{y}\| < 5$.
b) If $\|\mathbf{x}\| < 2$, $\|\mathbf{y}\| < 3$, and $\|\mathbf{z}\| < 4$, prove that $|\mathbf{x} \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{z}| < 14$.
c) If $\|\mathbf{x} - \mathbf{y}\| < 2$ and $\|\mathbf{z}\| < 3$, prove that $|\mathbf{x} \cdot (\mathbf{y} - \mathbf{z}) - \mathbf{y} \cdot (\mathbf{x} - \mathbf{z})| < 6$.
d) If $\|2\mathbf{x} - \mathbf{y}\| < 2$ and $\|\mathbf{y}\| < 1$, prove that $|\, \|\mathbf{x} - \mathbf{y}\|^2 - \mathbf{x} \cdot \mathbf{x}| < 2$.
e) If $n = 3$, $\|\mathbf{x} - \mathbf{y}\| < 2$, and $\|\mathbf{z}\| < 3$, prove that $\|\mathbf{x} \times \mathbf{z} - \mathbf{y} \times \mathbf{z}\| < 6$.
f) If $n = 3$, $\|\mathbf{x}\| < 1$, $\|\mathbf{y}\| < 2$, and $\|\mathbf{z}\| < 3$, prove that $\|\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})\| < 6$.

8.1.2. Let $B := \{\mathbf{x} \in \mathbf{R}^n : \|\mathbf{x}\| \le 1\}$.

a) If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in B$ and

$$\mathbf{v} := \frac{(\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\cdot\mathbf{c})\mathbf{b} + (\mathbf{c}\cdot\mathbf{b})\mathbf{a}}{3},$$

prove that $\mathbf{v}$ belongs to B.

b) If $\mathbf{a}, \mathbf{b} \in B$, prove that

$$|\mathbf{a}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{d}| \le \|\mathbf{b} - \mathbf{c}\| + \|\mathbf{a} - \mathbf{d}\|$$

for all $\mathbf{c}, \mathbf{d} \in \mathbf{R}^n$.

c) If $\mathbf{a}, \mathbf{b}, \mathbf{c} \in B$ and $n = 3$, prove that

$$\sqrt{|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|^2 + |\mathbf{a}\cdot\mathbf{b}|^2} \le 1.$$

8.1.3. Use the proof of Theorem 8.5 to show that equality in the Cauchy–Schwarz Inequality holds if and only if $\mathbf{x} = \mathbf{0}$, $\mathbf{y} = \mathbf{0}$, or $\mathbf{x}$ is parallel to $\mathbf{y}$.

8.1.4. Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero vectors in $\mathbf{R}^n$.

a) If $\phi(t) = \mathbf{a} + t\mathbf{b}$ for $t \in \mathbf{R}$, show that for each $t_0, t_1, t_2 \in \mathbf{R}$ with $t_1, t_2 \ne t_0$, the angle between $\phi(t_1) - \phi(t_0)$ and $\phi(t_2) - \phi(t_0)$ is 0 or π.

b) If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, show that $\mathbf{a}$ and $\mathbf{b}$ are parallel according to Definition 8.4 if and only if $\theta = 0$ or π, and that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal according to Definition 8.4 if and only if $\theta = \pi/2$.

8.1.5. The midpoint of a side of a triangle in $\mathbf{R}^3$ is the point that bisects that side (i.e., that divides it into two equal pieces). Let Δ be a triangle in $\mathbf{R}^3$ with sides A, B, and C and let L denote the line segment between the midpoints of A and B. Prove that L is parallel to C and that the length of L is one-half the length of C.

8.1.6. a) Prove that $(1, 2, 3)$, $(4, 5, 6)$, and $(0, 4, 2)$ are vertices of a right triangle in $\mathbf{R}^3$.

b) Find all nonzero vectors orthogonal to $(1, -1, 0)$ which lie in the plane $z = x$.

c) Find all nonzero vectors orthogonal to the vector $(3, 2, -5)$ whose components sum to 4.

8.1.7. Let $a < b$ be real numbers. The Cartesian product $[a, b] \times [a, b]$ is obviously a square in $\mathbf{R}^2$. Define a cube Q in $\mathbf{R}^n$ to be the n-fold Cartesian product of $[a, b]$ with itself; that is, $Q := [a, b] \times \cdots \times [a, b]$. Find a formula of the angle between the longest diagonal of Q and any of its edges. Show that when $n = 3$, this angle is approximately 54.74 degrees.

8.1.8. a) Using Postulate 1 in Section 1.2 and Definition 8.1, prove Theorem 8.2.

b) Prove Theorem 8.9, parts i) through iii) and vi).

c) Prove that if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$, then $\|\mathbf{x} \times \mathbf{y}\| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$.

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8.2 PLANES AND L

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8.1.9. Suppose that $\{a_k\}$ and $\{b_k\}$ are sequences of real numbers which satisfy

$$\sum_{k=1}^{\infty} a_k^2 < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} b_k^2 < \infty.$$

Prove that the infinite series $\sum_{k=1}^{\infty} a_k b_k$ converges absolutely.

8.1.10. Prove that the ℓ^1-norm and the sup-norm also satisfy Theorem 8.6.

8.2 PLANES AND LINEAR TRANSFORMATIONS

A plane Π in $\mathbf{R}^3$ is a set of points that is "flat" in some sense. What do we mean by flat? Any vector that lies in Π is orthogonal to a common direction, called the *normal*, which we will denote by $\mathbf{b}$. Fix a point $\mathbf{a} \in \Pi$. Since the vector $\mathbf{x} - \mathbf{a}$ lies in Π for all $\mathbf{x} \in \Pi$ and since two vectors are orthogonal when their dot product is zero, we see that $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ for all $\mathbf{x} \in \Pi$ (see Figure 8.4).

Using this three-dimensional case as a guide, for any $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ with $\mathbf{b} \neq \mathbf{0}$, we call the set

$$\Pi_{\mathbf{b}}(\mathbf{a}) := \{\mathbf{x} \in \mathbf{R}^n : (\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0\}$$

the *hyperplane* in $\mathbf{R}^n$ passing through a point $\mathbf{a} \in \mathbf{R}^n$ with normal $\mathbf{b}$. (We call it a *plane* when $n = 3$.) In particular, $\Pi_{\mathbf{b}}(\mathbf{a})$ is the set of all points $\mathbf{x}$ such that $\mathbf{x} - \mathbf{a}$ is orthogonal to $\mathbf{b}$.

There is nothing unique about "the normal" of a hyperplane: Any nonzero vector $\mathbf{c}$ parallel to $\mathbf{b}$ will define the same hyperplane. Indeed, if $\mathbf{b}$ and $\mathbf{c}$ are parallel, then, by definition, $\mathbf{b} = t\mathbf{c}$ for some nonzero $t \in \mathbf{R}$; hence $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{b} = 0$ if and only if $(\mathbf{x} - \mathbf{a}) \cdot \mathbf{c} = 0$. Nevertheless, many properties of hyperplane can be

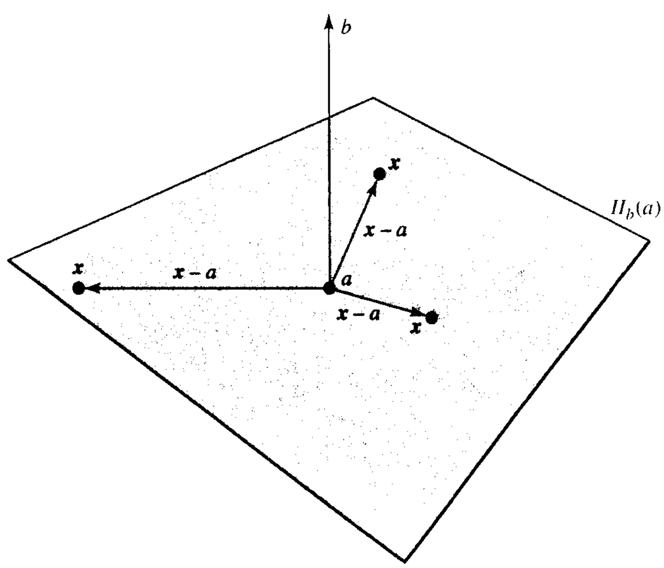

FIGURE 8.4

***8.18 Remark.** *If* $\mathbf{T} : \mathbf{R}^n \to \mathbf{R}^m$ *and* $\mathbf{U} : \mathbf{R}^m \to \mathbf{R}^p$ *are linear, then so is* $\mathbf{U} \circ \mathbf{T}$. *In fact, if B is the $m \times n$ matrix which represents* $\mathbf{T}$, *and C is the $p \times m$ matrix which represents* $\mathbf{U}$, *then CB is the matrix which represents* $\mathbf{U} \circ \mathbf{T}$.

Proof. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the usual basis of $\mathbf{R}^n$, $\mathbf{u}_1, \dots, \mathbf{u}_m$ be the usual basis of $\mathbf{R}^m$, and $\mathbf{w}_1, \dots, \mathbf{w}_p$ be the usual basis of $\mathbf{R}^p$. If $B = [b_{ij}]_{m \times n}$ represents $\mathbf{T}$ and $C = [c_{\nu k}]_{p \times m}$ represents $\mathbf{U}$, then, by Theorem 8.15,

$$\sum_{k=1}^{m} b_{kj}\mathbf{u}_k = (b_{1j}, \dots, b_{mj}) = \mathbf{T}(\mathbf{e}_j), \qquad j = 1, 2, \dots, n,$$

and

$$\sum_{\nu=1}^{p} c_{\nu k}\mathbf{w}_\nu = (c_{1k}, \dots, c_{pk}) = \mathbf{U}(\mathbf{u}_k), \qquad k = 1, 2, \dots, m.$$

Hence

$$\begin{aligned}(\mathbf{U} \circ \mathbf{T})(\mathbf{e}_j) = \mathbf{U}(\mathbf{T}(\mathbf{e}_j)) &= \mathbf{U}\left(\sum_{k=1}^{m} b_{kj}\mathbf{u}_k\right) = \sum_{k=1}^{m} b_{kj}\mathbf{U}(\mathbf{u}_k) \\ &= \sum_{k=1}^{m}\sum_{\nu=1}^{p} b_{kj}c_{\nu k}\mathbf{w}_\nu = \left(\sum_{k=1}^{m} b_{kj}c_{1k}, \dots, \sum_{k=1}^{m} b_{kj}c_{pk}\right)\end{aligned}$$

for each $1 \le j \le n$. Since this last vector is the jth column of the matrix CB, it follows that CB is the matrix which represents $\mathbf{U} \circ \mathbf{T}$. ∎

EXERCISES

8.2.1. Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}^3$.

a) Prove that if **a, b**, and **c** do not all lie on the same line, then an equation of the plane through these points is given by $(x, y, z) \cdot \mathbf{d} = \mathbf{a} \cdot \mathbf{d}$, where
$$\mathbf{d} := (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} - \mathbf{c}).$$

b) Prove that if **c** does not lie on the line $\phi(t) = t\mathbf{a} + \mathbf{b}$, $t \in \mathbf{R}$, then an equation of the plane that contains this line and the point **c** is given by $(x, y, z) \cdot \mathbf{d} = \mathbf{b} \cdot \mathbf{d}$, where $\mathbf{d} := \mathbf{a} \times (\mathbf{b} - \mathbf{c})$.

8.2.2. a) Find an equation of the hyperplane through the points $(1, 0, 0, 0)$, $(2, 1, 0, 0)$, $(0, 1, 1, 0)$, and $(0, 4, 0, 1)$.

b) Find an equation of the hyperplane that contains the lines $\phi(t) = (t, t, t, 1)$ and $\psi(t) = (1, t, 1 + t, t)$, $t \in \mathbf{R}$.

c) Find an equation of the plane parallel to the hyperplane $x_1 + \cdots + x_n = \pi$ passing through the point $(1, 2, \dots, n)$.

8.2.3. Find two lines in $\mathbf{R}^3$ which are not parallel but do not intersect.

8.2.4. Suppose that $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$ for some $n, m \in \mathbf{N}$.

a) Find the matrix representative of $\mathbf{T}$ if $\mathbf{T}(x, y, z, w) = (0, x + y, x - z, x + y + w)$.
b) Find the matrix representative of T if $T(x, y, z) = x - y + z$.
c) Find the matrix representative of $\mathbf{T}$ if $\mathbf{T}(x_1, x_2, \dots, x_n) = (x_1 - x_n, x_n - x_1)$.

8.2.5. Suppose that $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$ for some $n, m \in \mathbf{N}$.

a) If $\mathbf{T}(1, 1) = (3, \pi, 0)$ and $\mathbf{T}(0, 1) = (4, 0, 1)$, find the matrix representative of $\mathbf{T}$.
b) If $\mathbf{T}(1, 1, 0) = (e, \pi)$, $\mathbf{T}(0, -1, 1) = (1, 0)$, and $\mathbf{T}(1, 1, -1) = (1, 2)$, find the matrix representative of $\mathbf{T}$.
c) If $\mathbf{T}(0, 1, 1, 0) = (3, 5)$, $\mathbf{T}(0, 1, -1, 0) = (5, 3)$, and $\mathbf{T}(0, 0, 0, -1) = (\pi, 3)$, find all possible matrix representatives of $\mathbf{T}$.
d) If $\mathbf{T}(1, 1, 0, 0) = (5, 4, 1)$, $\mathbf{T}(0, 0, 1, 0) = (1, 2, 0)$, and $\mathbf{T}(0, 0, 0, -1) = (\pi, 3, -1)$, find all possible matrix representatives of $\mathbf{T}$.

8.2.6. Suppose that $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}^3$ are three points which do not lie on the same straight line and that Π is the plane which contains the points $\mathbf{a}, \mathbf{b}, \mathbf{c}$. Prove that an equation of Π is given by

$$\det \begin{bmatrix} x - a_1 & y - a_2 & z - a_3 \\ b_1 - a_1 & b_2 - a_2 & b_3 - a_3 \\ c_1 - a_1 & c_2 - a_2 & c_3 - a_3 \end{bmatrix} = 0.$$

$\boxed{\textbf{8.2.7}}$. **This exercise is used in Appendix E**. Recall that the area of a parallelogram with base b and altitude h is given by bh, and the volume of a parallelepiped is given by the area of its base times its altitude.

a) Let $\mathbf{a}, \mathbf{b} \in \mathbf{R}^3$ be nonzero vectors and $\mathcal{P}$ represent the parallelogram

$$\{(x, y, z) = u\mathbf{a} + v\mathbf{b} : u, v \in [0, 1]\}.$$

Prove that the area of $\mathcal{P}$ is $\|\mathbf{a} \times \mathbf{b}\|$.

b) Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{R}^3$ be nonzero vectors and $\mathcal{P}$ represent the parallelepiped

$$\{(x, y, z) = t\mathbf{a} + u\mathbf{b} + v\mathbf{c} : t, u, v \in [0, 1]\}.$$

Prove that the volume of $\mathcal{P}$ is $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$.

8.2.8. The distance from a point $\mathbf{x}_0 = (x_0, y_0, z_0)$ to a plane Π in $\mathbf{R}^3$ is defined to be

$$\operatorname{dist}(\mathbf{x}_0, \Pi) := \begin{cases} 0 & \mathbf{x}_0 \in \Pi \\ \|\mathbf{v}\| & \mathbf{x}_0 \notin \Pi, \end{cases}$$

where $\mathbf{v} := (x_0 - x_1, y_0 - y_1, z_0 - z_1)$ for some $(x_1, y_1, z_1) \in \Pi$, and $\mathbf{v}$ is orthogonal to Π (i.e., parallel to its normal). Sketch Π and $\mathbf{x}_0$ for a typical plane Π, and convince yourself that this is the correct definition. Prove that this definition does not depend on the choice of $\mathbf{v}$, by showing that the distance from $\mathbf{x}_0 = (x_0, y_0, z_0)$ to the plane Π described by $ax + by + cz = d$ is

$$\operatorname{dist}(\mathbf{x}_0, \Pi) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

$\boxed{\textbf{8.2.9}}$. [ROTATIONS IN $\mathbf{R}^2$]. **This exercise is used in Section *15.1**. Let

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some $\theta \in \mathbf{R}$.

a) Prove that $\|B(x, y)\| = \|(x, y)\|$ for all $(x, y) \in \mathbf{R}^2$.
b) Let $(x, y) \in \mathbf{R}^2$ be a nonzero vector and φ represent the angle between $B(x, y)$ and (x, y). Prove that $\cos\varphi = \cos\theta$. Thus, show that B rotates $\mathbf{R}^2$ through an angle θ. (When $\theta > 0$, we shall call B *counterclockwise rotation* about the origin through the angle θ.)

8.2.10. For each of the following functions $\mathbf{f}$, find the matrix representative of a linear transformation $\mathbf{T} \in \mathcal{L}(\mathbf{R}; \mathbf{R}^m)$ which satisfies

$$\lim_{h \to 0} \frac{\|\mathbf{f}(x + h) - \mathbf{f}(x) - \mathbf{T}(h)\|}{h} = 0.$$

a) $\mathbf{f}(x) = (x^2, \sin x)$
b) $\mathbf{f}(x) = (e^x, \sqrt[3]{x}, 1 - x^2)$
c) $\mathbf{f}(x) = (1, 2, 3, x^2 + x, x^2 - x)$

8.2.11. Fix $\mathbf{T} \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^m)$. Set

$$M_1 := \sup_{\|\mathbf{x}\|=1} \|\mathbf{T}(\mathbf{x})\| \quad \text{and}$$
$$M_2 := \inf\{C > 0 : \|\mathbf{T}(\mathbf{x})\| \le C\|\mathbf{x}\| \text{ for all } \mathbf{x} \in \mathbf{R}^n\}.$$

a) Prove that $M_1 \le \|\mathbf{T}\|$.
b) Using the linear property of $\mathbf{T}$, prove that if $\mathbf{x} \ne \mathbf{0}$, then

$$\frac{\|\mathbf{T}(\mathbf{x})\|}{\|\mathbf{x}\|} \le M_1.$$

c) Prove that $M_1 = M_2 = \|\mathbf{T}\|$.