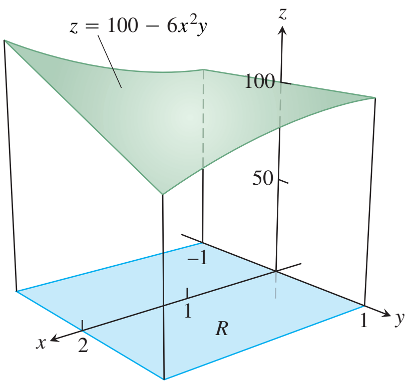


FIGURE 15.6 The double integral $\iint_R f(x, y)\,dA$ gives the volume under this surface over the rectangular region R (Example 1).

Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\iint_R f(x, y)\,dA = \int_{-1}^{1}\int_0^2 (100 - 6x^2y)\,dx\,dy = \int_{-1}^{1}\left[100x - 2x^3y\right]_{x=0}^{x=2} dy$$

$$= \int_{-1}^{1} (200 - 16y)\,dy = \left[200y - 8y^2\right]_{-1}^{1} = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1} (100 - 6x^2y)\,dy\,dx = \int_0^2 \left[100y - 3x^2y^2\right]_{y=-1}^{y=1} dx$$

$$= \int_0^2 [(100 - 3x^2) - (-100 - 3x^2)]\,dx$$

$$= \int_0^2 200\,dx = 400.$$

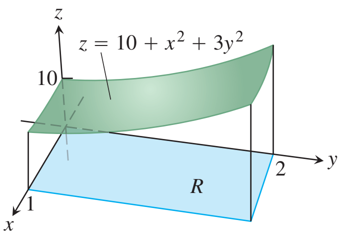


FIGURE 15.7 The double integral $\iint_R f(x, y)\,dA$ gives the volume under this surface over the rectangular region R (Example 2).

EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1$, $0 \le y \le 2$.

Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\,dA = \int_0^1\int_0^2 (10 + x^2 + 3y^2)\,dy\,dx$$

$$= \int_0^1 \left[10y + x^2y + y^3\right]_{y=0}^{y=2} dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\,dx = \left[20x + \frac{2}{3}x^3 + 8x\right]_0^1 = \frac{86}{3}.$$

Exercises 15.1

Evaluating Iterated Integrals

In Exercises 1–12, evaluate the iterated integral.

1. $\int_1^2\int_0^4 2xy\,dy\,dx$
2. $\int_0^2\int_{-1}^{1} (x - y)\,dy\,dx$
3. $\int_{-1}^{0}\int_{-1}^{1} (x + y + 1)\,dx\,dy$
4. $\int_0^1\int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right) dx\,dy$
5. $\int_0^3\int_0^2 (4 - y^2)\,dy\,dx$
6. $\int_0^3\int_{-2}^{0} (x^2y - 2xy)\,dy\,dx$
7. $\int_0^1\int_0^1 \frac{y}{1 + xy}\,dx\,dy$
8. $\int_1^4\int_0^4 \left(\frac{x}{2} + \sqrt{y}\right) dx\,dy$
9. $\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx$
10. $\int_0^1\int_1^2 xye^x\,dy\,dx$
11. $\int_{-1}^{2}\int_0^{\pi/2} y\sin x\,dx\,dy$
12. $\int_{\pi}^{2\pi}\int_0^{\pi} (\sin x + \cos y)\,dx\,dy$

Evaluating Double Integrals over Rectangles

In Exercises 13–20, evaluate the double integral over the given region R.

13. $\iint_R (6y^2 - 2x)\,dA$, $\quad R$: $0 \le x \le 1$, $0 \le y \le 2$
14. $\iint_R \left(\frac{\sqrt{x}}{y^2}\right) dA$, $\quad R$: $0 \le x \le 4$, $1 \le y \le 2$
15. $\iint_R xy\cos y\,dA$, $\quad R$: $-1 \le x \le 1$, $0 \le y \le \pi$

16. $\displaystyle\iint_R y \sin(x + y)\, dA, \quad R: \; -\pi \le x \le 0, \quad 0 \le y \le \pi$

17. $\displaystyle\iint_R e^{x-y}\, dA, \quad R: \; 0 \le x \le \ln 2, \quad 0 \le y \le \ln 2$

18. $\displaystyle\iint_R xye^{xy^2}\, dA, \quad R: \; 0 \le x \le 2, \quad 0 \le y \le 1$

19. $\displaystyle\iint_R \frac{xy^3}{x^2 + 1}\, dA, \quad R: \; 0 \le x \le 1, \quad 0 \le y \le 2$

20. $\displaystyle\iint_R \frac{y}{x^2y^2 + 1}\, dA, \quad R: \; 0 \le x \le 1, \quad 0 \le y \le 1$

In Exercises 21 and 22, integrate f over the given region.

21. **Square** $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2, 1 \le y \le 2$

22. **Rectangle** $f(x, y) = y \cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

Volume Beneath a Surface $z = f(x, y)$

23. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1$, $-1 \le y \le 1$.

24. Find the volume of the region bounded above by the elliptical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: 0 \le x \le 2, 0 \le y \le 2$.

25. Find the volume of the region bounded above by the plane $z = 2 - x - y$ and below by the square $R: 0 \le x \le 1$, $0 \le y \le 1$.

26. Find the volume of the region bounded above by the plane $z = y/2$ and below by the rectangle $R: 0 \le x \le 4, 0 \le y \le 2$.

27. Find the volume of the region bounded above by the surface $z = 2 \sin x \cos y$ and below by the rectangle $R: 0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

28. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R: 0 \le x \le 1$, $0 \le y \le 2$.

15.2 Double Integrals over General Regions

In this section we define and evaluate double integrals over bounded regions in the plane which are more general than rectangles. These double integrals are also evaluated as iterated integrals, with the main practical problem being that of determining the limits of integration. Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the limits of integration often involve variables, not just constants.

Double Integrals over Bounded, Nonrectangular Regions

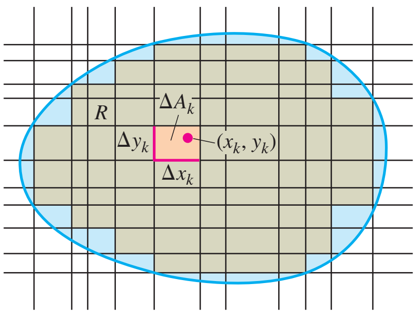


FIGURE 15.8 A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R, such as the one in Figure 15.8, we again begin by covering R with a grid of small rectangular cells whose union contains all points of R. This time, however, we cannot exactly fill R with a finite number of rectangles lying inside R, since its boundary is curved, and some of the small rectangles in the grid lie partly outside R. A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.

Once we have a partition of R, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k.$$

As the norm of the partition forming S_n goes to zero, $\|P\| \to 0$, the width and height of each enclosed rectangle goes to zero and their number goes to infinity. If $f(x, y)$ is a continuous function, then these Riemann sums converge to a limiting value, not dependent on any of the choices we made. This limit is called the **double integral** of $f(x, y)$ over R:

$$\lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k = \iint_R f(x, y)\, dA.$$

The idea behind these properties is that integrals behave like sums. If the function $f(x, y)$ is replaced by its constant multiple $cf(x, y)$, then a Riemann sum for f

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

is replaced by a Riemann sum for cf

$$\sum_{k=1}^{n} cf(x_k, y_k)\, \Delta A_k = c \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k = cS_n .$$

Taking limits as $n \to \infty$ shows that $c \lim_{n\to\infty} S_n = c \iint_R f\, dA$ and $\lim_{n\to\infty} cS_n = \iint_R cf\, dA$ are equal. It follows that the constant multiple property carries over from sums to double integrals.

The other properties are also easy to verify for Riemann sums, and carry over to double integrals for the same reason. While this discussion gives the idea, an actual proof that these properties hold requires a more careful analysis of how Riemann sums converge.

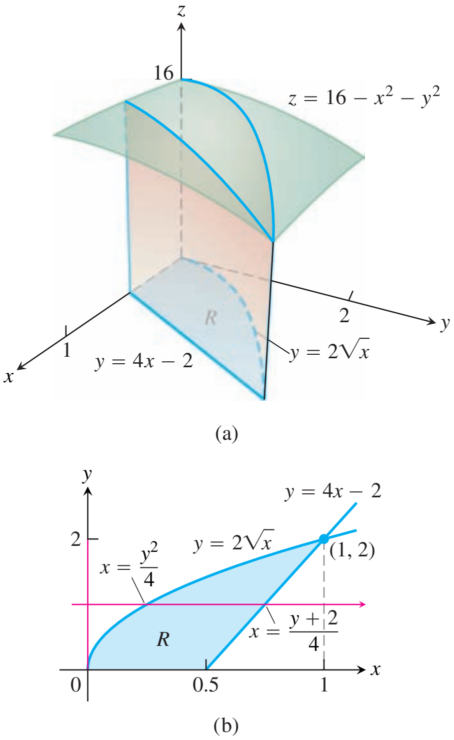


FIGURE 15.18 (a) The solid "wedgelike" region whose volume is found in Example 4. (b) The region of integration R showing the order $dx\,dy$.

EXAMPLE 4 Find the volume of the wedgelike solid that lies beneath the surface $z = 16 - x^2 - y^2$ and above the region R bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the x-axis.

Solution Figure 15.18a shows the surface and the "wedgelike" solid whose volume we want to calculate. Figure 15.18b shows the region of integration in the xy-plane. If we integrate in the order $dy\,dx$ (first with respect to y and then with respect to x), two integrations will be required because y varies from $y = 0$ to $y = 2\sqrt{x}$ for $0 \le x \le 0.5$, and then varies from $y = 4x - 2$ to $y = 2\sqrt{x}$ for $0.5 \le x \le 1$. So we choose to integrate in the order $dx\,dy$, which requires only one double integral whose limits of integration are indicated in Figure 15.18b. The volume is then calculated as the iterated integral:

$$\begin{aligned}
&\iint_R (16 - x^2 - y^2)\, dA \\
&= \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx\, dy \\
&= \int_0^2 \left[16x - \frac{x^3}{3} - xy^2\right]_{x=y^2/4}^{x=(y+2)/4} dx \\
&= \int_0^2 \left[4(y+2) - \frac{(y+2)^3}{3\cdot 64} - \frac{(y+2)y^2}{4} - 4y^2 + \frac{y^6}{3\cdot 64} + \frac{y^4}{4}\right] dy \\
&= \left[\frac{191y}{24} + \frac{63y^2}{32} - \frac{145y^3}{96} - \frac{49y^4}{768} + \frac{y^5}{20} + \frac{y^7}{1344}\right]_0^2 = \frac{20803}{1680} \approx 12.4.
\end{aligned}$$

■

Exercises 15.2

Sketching Regions of Integration

In Exercises 1–8, sketch the described regions of integration.

1. $0 \le x \le 3, \quad 0 \le y \le 2x$

2. $-1 \le x \le 2, \quad x - 1 \le y \le x^2$

3. $-2 \le y \le 2, \quad y^2 \le x \le 4$

4. $0 \le y \le 1, \quad y \le x \le 2y$

5. $0 \le x \le 1, \quad e^x \le y \le e$

6. $1 \le x \le e^2, \quad 0 \le y \le \ln x$

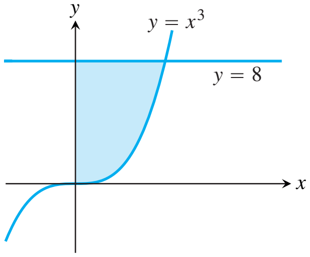
7. $0 \le y \le 1, \quad 0 \le x \le \sin^{-1} y$

8. $0 \le y \le 8, \quad \frac{1}{4}y \le x \le y^{1/3}$

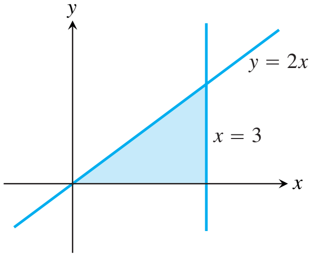
Finding Limits of Integration

In Exercises 9–18, write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

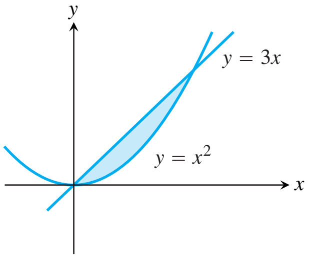
9.



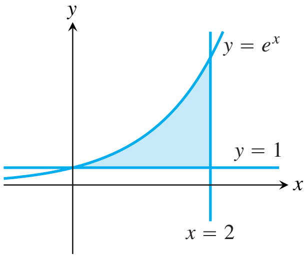
10.



11.



12.



13. Bounded by $y = \sqrt{x}$, $y = 0$, and $x = 9$

14. Bounded by $y = \tan x$, $x = 0$, and $y = 1$

15. Bounded by $y = e^{-x}$, $y = 1$, and $x = \ln 3$

16. Bounded by $y = 0$, $x = 0$, $y = 1$, and $y = \ln x$

17. Bounded by $y = 3 - 2x$, $y = x$, and $x = 0$

18. Bounded by $y = x^2$ and $y = x + 2$

Finding Regions of Integration and Double Integrals

In Exercises 19–24, sketch the region of integration and evaluate the integral.

19. $\displaystyle\int_0^{\pi}\int_0^{x} x \sin y \, dy\, dx$

20. $\displaystyle\int_0^{\pi}\int_0^{\sin x} y \, dy\, dx$

21. $\displaystyle\int_1^{\ln 8}\int_0^{\ln y} e^{x+y} \, dx\, dy$

22. $\displaystyle\int_1^{2}\int_y^{y^2} dx\, dy$

23. $\displaystyle\int_0^{1}\int_0^{y^2} 3y^3 e^{xy} \, dx\, dy$

24. $\displaystyle\int_1^{4}\int_0^{\sqrt{x}} \frac{3}{2} e^{y/\sqrt{x}} \, dy\, dx$

In Exercises 25–28, integrate f over the given region.

25. Quadrilateral $f(x, y) = x/y$ over the region in the first quadrant bounded by the lines $y = x$, $y = 2x$, $x = 1$, and $x = 2$

26. Triangle $f(x, y) = x^2 + y^2$ over the triangular region with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$

27. Triangle $f(u, v) = v - \sqrt{u}$ over the triangular region cut from the first quadrant of the uv-plane by the line $u + v = 1$

28. Curved region $f(s, t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$

Each of Exercises 29–32 gives an integral over a region in a Cartesian coordinate plane. Sketch the region and evaluate the integral.

29. $\displaystyle\int_{-2}^{0}\int_{v}^{-v} 2 \, dp\, dv$ (the pv-plane)

30. $\displaystyle\int_0^{1}\int_0^{\sqrt{1-s^2}} 8t \, dt\, ds$ (the st-plane)

31. $\displaystyle\int_{-\pi/3}^{\pi/3}\int_0^{\sec t} 3 \cos t \, du\, dt$ (the tu-plane)

32. $\displaystyle\int_0^{3/2}\int_1^{4-2u} \frac{4 - 2u}{v^2} \, dv\, du$ (the uv-plane)

Reversing the Order of Integration

In Exercises 33–46, sketch the region of integration and write an equivalent double integral with the order of integration reversed.

33. $\displaystyle\int_0^{1}\int_2^{4-2x} dy\, dx$

34. $\displaystyle\int_0^{2}\int_{y-2}^{0} dx\, dy$

35. $\displaystyle\int_0^{1}\int_y^{\sqrt{y}} dx\, dy$

36. $\displaystyle\int_0^{1}\int_{1-x}^{1-x^2} dy\, dx$

37. $\displaystyle\int_0^{1}\int_1^{e^x} dy\, dx$

38. $\displaystyle\int_0^{\ln 2}\int_{e^y}^{2} dx\, dy$

39. $\displaystyle\int_0^{3/2}\int_0^{9-4x^2} 16x \, dy\, dx$

40. $\displaystyle\int_0^{2}\int_0^{4-y^2} y \, dx\, dy$

41. $\displaystyle\int_0^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y \, dx\, dy$

42. $\displaystyle\int_0^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 6x \, dy\, dx$

43. $\displaystyle\int_1^{e}\int_0^{\ln x} xy \, dy\, dx$

44. $\displaystyle\int_0^{\pi/6}\int_{\sin x}^{1/2} xy^2 \, dy\, dx$

45. $\displaystyle\int_0^{3}\int_1^{e^y} (x + y) \, dx\, dy$

46. $\displaystyle\int_0^{\sqrt{3}}\int_0^{\tan^{-1} y} \sqrt{xy} \, dx\, dy$

In Exercises 47–56, sketch the region of integration, reverse the order of integration, and evaluate the integral.

47. $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y} \, dy\, dx$

48. $\displaystyle\int_0^{2}\int_x^{2} 2y^2 \sin xy \, dy\, dx$

49. $\displaystyle\int_0^{1}\int_y^{1} x^2 e^{xy} \, dx\, dy$

50. $\displaystyle\int_0^{2}\int_0^{4-x^2} \frac{xe^{2y}}{4 - y} \, dy\, dx$

51. $\displaystyle\int_0^{2\sqrt{\ln 3}}\int_{y/2}^{\sqrt{\ln 3}} e^{x^2} \, dx\, dy$

52. $\displaystyle\int_0^{3}\int_{\sqrt{x/3}}^{1} e^{y^3} \, dy\, dx$

53. $\displaystyle\int_0^{1/16}\int_{y^{1/4}}^{1/2} \cos(16\pi x^5) \, dx\, dy$

54. $\displaystyle\int_0^{8}\int_{\sqrt[3]{x}}^{2} \frac{dy\, dx}{y^4 + 1}$

55. Square region $\iint_R (y - 2x^2)\, dA$ where R is the region bounded by the square $|x| + |y| = 1$

56. Triangular region $\iint_R xy \, dA$ where R is the region bounded by the lines $y = x$, $y = 2x$, and $x + y = 2$

Volume Beneath a Surface $z = f(x, y)$

57. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

58. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

59. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

60. Find the volume of the solid in the first octant bounded by the coordinate planes, the cylinder $x^2 + y^2 = 4$, and the plane $z + y = 3$.

61. Find the volume of the solid in the first octant bounded by the coordinate planes, the plane $x = 3$, and the parabolic cylinder $z = 4 - y^2$.

62. Find the volume of the solid cut from the first octant by the surface $z = 4 - x^2 - y$.

63. Find the volume of the wedge cut from the first octant by the cylinder $z = 12 - 3y^2$ and the plane $x + y = 2$.

64. Find the volume of the solid cut from the square column $|x| + |y| \le 1$ by the planes $z = 0$ and $3x + z = 3$.

65. Find the volume of the solid that is bounded on the front and back by the planes $x = 2$ and $x = 1$, on the sides by the cylinders $y = \pm 1/x$, and above and below by the planes $z = x + 1$ and $z = 0$.

66. Find the volume of the solid bounded on the front and back by the planes $x = \pm\pi/3$, on the sides by the cylinders $y = \pm \sec x$, above by the cylinder $z = 1 + y^2$, and below by the xy-plane.

In Exercises 67 and 68, sketch the region of integration and the solid whose volume is given by the double integral.

67. $\displaystyle\int_0^3 \int_0^{2-2x/3} \left(1 - \frac{1}{3}x - \frac{1}{2}y\right) dy\,dx$

68. $\displaystyle\int_0^4 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \sqrt{25 - x^2 - y^2}\,dx\,dy$

Integrals over Unbounded Regions

Improper double integrals can often be computed similarly to improper integrals of one variable. The first iteration of the following improper integrals is conducted just as if they were proper integrals. One then evaluates an improper integral of a single variable by taking appropriate limits, as in Section 8.7. Evaluate the improper integrals in Exercises 69–72 as iterated integrals.

69. $\displaystyle\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y}\,dy\,dx$

70. $\displaystyle\int_{-1}^1 \int_{-1/\sqrt{1-x^2}}^{1/\sqrt{1-x^2}} (2y + 1)\,dy\,dx$

71. $\displaystyle\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{(x^2 + 1)(y^2 + 1)}\,dx\,dy$

72. $\displaystyle\int_0^\infty \int_0^\infty xe^{-(x+2y)}\,dx\,dy$

Approximating Integrals with Finite Sums

In Exercises 73 and 74, approximate the double integral of $f(x, y)$ over the region R partitioned by the given vertical lines $x = a$ and horizontal lines $y = c$. In each subrectangle, use (x_k, y_k) as indicated for your approximation.

$$\iint_R f(x, y)\,dA \approx \sum_{k=1}^n f(x_k, y_k)\,\Delta A_k$$

73. $f(x, y) = x + y$ over the region R bounded above by the semicircle $y = \sqrt{1 - x^2}$ and below by the x-axis, using the partition $x = -1, -1/2, 0, 1/4, 1/2, 1$ and $y = 0, 1/2, 1$ with (x_k, y_k) the lower left corner in the kth subrectangle (provided the subrectangle lies within R)

74. $f(x, y) = x + 2y$ over the region R inside the circle $(x - 2)^2 + (y - 3)^2 = 1$ using the partition $x = 1, 3/2, 2, 5/2, 3$ and $y = 2, 5/2, 3, 7/2, 4$ with (x_k, y_k) the center (centroid) in the kth subrectangle (provided the subrectangle lies within R)

Theory and Examples

75. Circular sector Integrate $f(x, y) = \sqrt{4 - x^2}$ over the smaller sector cut from the disk $x^2 + y^2 \le 4$ by the rays $\theta = \pi/6$ and $\theta = \pi/2$.

76. Unbounded region Integrate $f(x, y) = 1/[(x^2 - x)(y - 1)^{2/3}]$ over the infinite rectangle $2 \le x < \infty, 0 \le y \le 2$.

77. Noncircular cylinder A solid right (noncircular) cylinder has its base R in the xy-plane and is bounded above by the paraboloid $z = x^2 + y^2$. The cylinder's volume is

$$V = \int_0^1 \int_0^y (x^2 + y^2)\,dx\,dy + \int_1^2 \int_0^{2-y} (x^2 + y^2)\,dx\,dy.$$

Sketch the base region R and express the cylinder's volume as a single iterated integral with the order of integration reversed. Then evaluate the integral to find the volume.

78. Converting to a double integral Evaluate the integral

$$\int_0^2 (\tan^{-1}\pi x - \tan^{-1} x)\,dx.$$

(*Hint:* Write the integrand as an integral.)

79. Maximizing a double integral What region R in the xy-plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2)\,dA?$$

Give reasons for your answer.

80. Minimizing a double integral What region R in the xy-plane minimizes the value of

$$\iint_R (x^2 + y^2 - 9)\,dA?$$

Give reasons for your answer.

81. Is it possible to evaluate the integral of a continuous function $f(x, y)$ over a rectangular region in the xy-plane and get different answers depending on the order of integration? Give reasons for your answer.

82 How would you evaluate the double integral of a continuous function $f(x, y)$ over the region R in the xy-plane enclosed by the triangle with vertices (0, 1), (2, 0), and (1, 2)? Give reasons for your answer.

83. Unbounded region Prove that

$$\int_{-\infty}^\infty \int_{-\infty}^\infty e^{-x^2-y^2}\,dx\,dy = \lim_{b\to\infty} \int_{-b}^b \int_{-b}^b e^{-x^2-y^2}\,dx\,dy = 4\left(\int_0^\infty e^{-x^2}\,dx\right)^2.$$

84. Improper double integral Evaluate the improper integral

$$\int_0^1 \int_0^3 \frac{x^2}{(y-1)^{2/3}}\,dy\,dx.$$

COMPUTER EXPLORATIONS

Use a CAS double-integral evaluator to estimate the values of the integrals in Exercises 85–88.

85. $\displaystyle\int_1^3 \int_1^x \frac{1}{xy}\,dy\,dx$

86. $\displaystyle\int_0^1 \int_0^1 e^{-(x^2+y^2)}\,dy\,dx$

87. $\displaystyle\int_0^1 \int_0^1 \tan^{-1} xy\,dy\,dx$

88. $\displaystyle\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3\sqrt{1-x^2-y^2}\,dy\,dx$

Use a CAS double-integral evaluator to find the integrals in Exercises 89–94. Then reverse the order of integration and evaluate, again with a CAS.

89. $\displaystyle\int_0^1 \int_{2y}^4 e^{x^2}\,dx\,dy$

90. $\displaystyle\int_0^3 \int_{x^2}^9 x\cos(y^2)\,dy\,dx$

91. $\displaystyle\int_0^2 \int_{y^3}^{4\sqrt{2y}} (x^2y - xy^2)\,dx\,dy$

92. $\displaystyle\int_0^2 \int_0^{4-y^2} e^{xy}\,dx\,dy$

93. $\displaystyle\int_1^2 \int_0^{x^2} \frac{1}{x+y}\,dy\,dx$

94. $\displaystyle\int_1^2 \int_{y^3}^8 \frac{1}{\sqrt{x^2+y^2}}\,dx\,dy$

15.3 Area by Double Integration

In this section we show how to use double integrals to calculate the areas of bounded regions in the plane, and to find the average value of a function of two variables.

Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R in the preceding section, the Riemann sums reduce to

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k = \sum_{k=1}^{n} \Delta A_k. \tag{1}$$

This is simply the sum of the areas of the small rectangles in the partition of R, and approximates what we would like to call the area of R. As the norm of a partition of R approaches zero, the height and width of all rectangles in the partition approach zero, and the coverage of R becomes increasingly complete (Figure 15.8). We define the area of R to be the limit

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} \Delta A_k = \iint_R dA. \tag{2}$$

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

As with the other definitions in this chapter, the definition here applies to a greater variety of regions than does the earlier single-variable definition of area, but it agrees with the earlier definition on regions to which they both apply. To evaluate the integral in the definition of area, we integrate the constant function $f(x, y) = 1$ over R.

EXAMPLE 1 Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.

EXAMPLE 3 Find the average value of $f(x, y) = x\cos xy$ over the rectangle $R: 0 \le x \le \pi,\ 0 \le y \le 1$.

Solution The value of the integral of f over R is

$$\int_0^\pi \int_0^1 x\cos xy\, dy\, dx = \int_0^\pi \Big[\sin xy\Big]_{y=0}^{y=1} dx \qquad \int x\cos xy\, dy = \sin xy + C$$

$$= \int_0^\pi (\sin x - 0)\, dx = -\cos x\Big]_0^\pi = 1 + 1 = 2.$$

The area of R is π. The average value of f over R is $2/\pi$. ■

Exercises 15.3

Area by Double Integrals

In Exercises 1–12, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

1. The coordinate axes and the line $x + y = 2$
2. The lines $x = 0$, $y = 2x$, and $y = 4$
3. The parabola $x = -y^2$ and the line $y = x + 2$
4. The parabola $x = y - y^2$ and the line $y = -x$
5. The curve $y = e^x$ and the lines $y = 0$, $x = 0$, and $x = \ln 2$
6. The curves $y = \ln x$ and $y = 2\ln x$ and the line $x = e$, in the first quadrant
7. The parabolas $x = y^2$ and $x = 2y - y^2$
8. The parabolas $x = y^2 - 1$ and $x = 2y^2 - 2$
9. The lines $y = x$, $y = x/3$, and $y = 2$
10. The lines $y = 1 - x$ and $y = 2$ and the curve $y = e^x$
11. The lines $y = 2x$, $y = x/2$, and $y = 3 - x$
12. The lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$

Identifying the Region of Integration

The integrals and sums of integrals in Exercises 13–18 give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

13. $\displaystyle\int_0^6 \int_{y^2/3}^{2y} dx\, dy$

14. $\displaystyle\int_0^3 \int_{-x}^{x(2-x)} dy\, dx$

15. $\displaystyle\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\, dx$

16. $\displaystyle\int_{-1}^2 \int_{y^2}^{y+2} dx\, dy$

17. $\displaystyle\int_{-1}^0 \int_{-2x}^{1-x} dy\, dx + \int_0^2 \int_{-x/2}^{1-x} dy\, dx$

18. $\displaystyle\int_0^2 \int_{x^2-4}^0 dy\, dx + \int_0^4 \int_0^{\sqrt{x}} dy\, dx$

Finding Average Values

19. Find the average value of $f(x, y) = \sin(x + y)$ over
    a. the rectangle $0 \le x \le \pi,\quad 0 \le y \le \pi$.
    b. the rectangle $0 \le x \le \pi,\quad 0 \le y \le \pi/2$.
20. Which do you think will be larger, the average value of $f(x, y) = xy$ over the square $0 \le x \le 1$, $0 \le y \le 1$, or the average value of f over the quarter circle $x^2 + y^2 \le 1$ in the first quadrant? Calculate them to find out.
21. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.
22. Find the average value of $f(x, y) = 1/(xy)$ over the square $\ln 2 \le x \le 2\ln 2$, $\ln 2 \le y \le 2\ln 2$.

Theory and Examples

23. **Bacterium population** If $f(x, y) = (10{,}000e^y)/(1 + |x|/2)$ represents the "population density" of a certain bacterium on the xy-plane, where x and y are measured in centimeters, find the total population of bacteria within the rectangle $-5 \le x \le 5$ and $-2 \le y \le 0$.
24. **Regional population** If $f(x, y) = 100\,(y + 1)$ represents the population density of a planar region on Earth, where x and y are measured in miles, find the number of people in the region bounded by the curves $x = y^2$ and $x = 2y - y^2$.
25. **Average temperature in Texas** According to the *Texas Almanac*, Texas has 254 counties and a National Weather Service station in each county. Assume that at time t_0, each of the 254 weather stations recorded the local temperature. Find a formula that would give a reasonable approximation of the average temperature in Texas at time t_0. Your answer should involve information that you would expect to be readily available in the *Texas Almanac*.
26. If $y = f(x)$ is a nonnegative continuous function over the closed interval $a \le x \le b$, show that the double integral definition of area for the closed plane region bounded by the graph of f, the vertical lines $x = a$ and $x = b$, and the x-axis agrees with the definition for area beneath the curve in Section 5.3.

$$\iint_R (9 - x^2 - y^2)\, dA = \int_0^{2\pi}\int_0^1 (9 - r^2)\, r\, dr\, d\theta$$
$$= \int_0^{2\pi}\int_0^1 (9r - r^3)\, dr\, d\theta$$
$$= \int_0^{2\pi}\left[\frac{9}{2}r^2 - \frac{1}{4}r^4\right]_{r=0}^{r=1} d\theta$$
$$= \frac{17}{4}\int_0^{2\pi} d\theta = \frac{17\pi}{2}.$$

EXAMPLE 6 Using polar integration, find the area of the region R in the xy-plane enclosed by the circle $x^2 + y^2 + 4$, above the line $y = 1$, and below the line $y = \sqrt{3}x$.

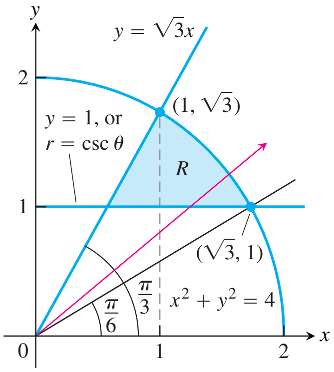


FIGURE 15.28 The region R in Example 6.

Solution A sketch of the region R is shown in Figure 15.28. First we note that the line $y = \sqrt{3}x$ has slope $\sqrt{3} = \tan\theta$, so $\theta = \pi/3$. Next we observe that the line $y = 1$ intersects the circle $x^2 + y^2 = 4$ when $x^2 + 1 = 4$, or $x = \sqrt{3}$. Moreover, the radial line from the origin through the point $(\sqrt{3}, 1)$ has slope $1/\sqrt{3} = \tan\theta$, giving its angle of inclination as $\theta = \pi/6$. This information is shown in Figure 15.28.

Now, for the region R, as θ varies from $\pi/6$ to $\pi/3$, the polar coordinate r varies from the horizontal line $y = 1$ to the circle $x^2 + y^2 = 4$. Substituting $r\sin\theta$ for y in the equation for the horizontal line, we have $r\sin\theta = 1$, or $r = \csc\theta$, which is the polar equation of the line. The polar equation for the circle is $r = 2$. So in polar coordinates, for $\pi/6 \le \theta \le \pi/3$, r varies from $r = \csc\theta$ to $r = 2$. It follows that the iterated integral for the area then gives

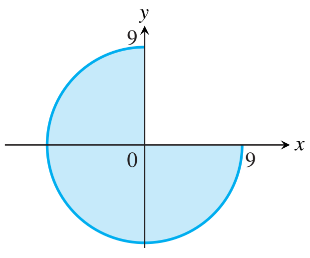
$$\iint_R dA = \int_{\pi/6}^{\pi/3}\int_{\csc\theta}^{2} r\, dr\, d\theta$$
$$= \int_{\pi/6}^{\pi/3}\left[\frac{1}{2}r^2\right]_{r=\csc\theta}^{r=2} d\theta$$
$$= \int_{\pi/6}^{\pi/3}\frac{1}{2}\left[4 - \csc^2\theta\right] d\theta$$
$$= \frac{1}{2}\left[4\theta + \cot\theta\right]_{\pi/6}^{\pi/3}$$
$$= \frac{1}{2}\left(\frac{4\pi}{3} + \frac{1}{\sqrt{3}}\right) - \frac{1}{2}\left(\frac{4\pi}{6} + \sqrt{3}\right) = \frac{\pi - \sqrt{3}}{3}.$$

Exercises 15.4

Regions in Polar Coordinates

In Exercises 1–8, describe the given region in polar coordinates.

1.



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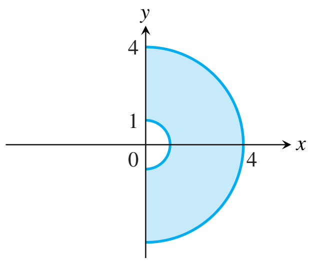
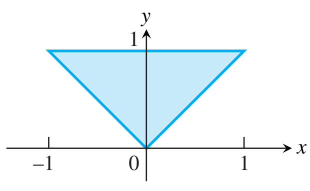
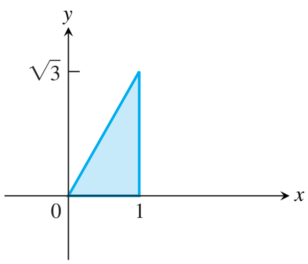


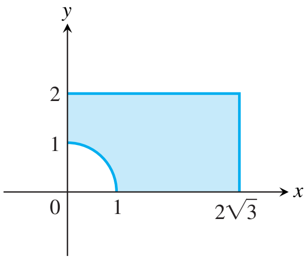
3.



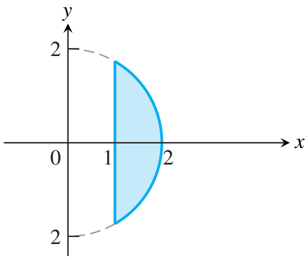
4.



5.



6.



7. The region enclosed by the circle $x^2 + y^2 = 2x$.

8. The region enclosed by the semicircle $x^2 + y^2 = 2y$, $y \geq 0$.

Evaluating Polar Integrals

In Exercises 9–22, change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

9. $\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} dy\, dx$

10. $\int_{0}^{1}\int_{0}^{\sqrt{1-y^2}} (x^2 + y^2)\, dx\, dy$

11. $\int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} (x^2 + y^2)\, dx\, dy$

12. $\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$

13. $\int_{0}^{6}\int_{0}^{y} x\, dx\, dy$

14. $\int_{0}^{2}\int_{0}^{x} y\, dy\, dx$

15. $\int_{1}^{\sqrt{3}}\int_{1}^{x} dy\, dx$

16. $\int_{\sqrt{2}}^{2}\int_{\sqrt{4-y^2}}^{y} dx\, dy$

17. $\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \dfrac{2}{1 + \sqrt{x^2 + y^2}}\, dy\, dx$

18. $\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \dfrac{2}{(1 + x^2 + y^2)^2}\, dy\, dx$

19. $\int_{0}^{\ln 2}\int_{0}^{\sqrt{(\ln 2)^2-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$

20. $\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\, dx\, dy$

21. $\int_{0}^{1}\int_{x}^{\sqrt{2-x^2}} (x + 2y)\, dy\, dx$

22. $\int_{1}^{2}\int_{0}^{\sqrt{2x-x^2}} \dfrac{1}{(x^2 + y^2)^2}\, dy\, dx$

In Exercises 23–26, sketch the region of integration and convert each polar integral or sum of integrals to a Cartesian integral or sum of integrals. Do not evaluate the integrals.

23. $\int_{0}^{\pi/2}\int_{0}^{1} r^3 \sin\theta \cos\theta\, dr\, d\theta$

24. $\int_{\pi/6}^{\pi/2}\int_{1}^{\csc\theta} r^2 \cos\theta\, dr\, d\theta$

25. $\int_{0}^{\pi/4}\int_{0}^{2\sec\theta} r^5 \sin^2\theta\, dr\, d\theta$

26. $\int_{0}^{\tan^{-1}\frac{4}{3}}\int_{0}^{3\sec\theta} r^7\, dr\, d\theta + \int_{\tan^{-1}\frac{4}{3}}^{\pi/2}\int_{0}^{4\csc\theta} r^7\, dr\, d\theta$

Area in Polar Coordinates

27. Find the area of the region cut from the first quadrant by the curve $r = 2(2 - \sin 2\theta)^{1/2}$.

28. Cardioid overlapping a circle Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

29. One leaf of a rose Find the area enclosed by one leaf of the rose $r = 12\cos 3\theta$.

30. Snail shell Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3$, $0 \leq \theta \leq 2\pi$. The region looks like a snail shell.

31. Cardioid in the first quadrant Find the area of the region cut from the first quadrant by the cardioid $r = 1 + \sin\theta$.

32. Overlapping cardioids Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

Average values

In polar coordinates, the **average value** of a function over a region R (Section 15.3) is given by

$$\frac{1}{\text{Area}(R)}\iint_R f(r, \theta)\, r\, dr\, d\theta.$$

33. Average height of a hemisphere Find the average height of the hemispherical surface $z = \sqrt{a^2 - x^2 - y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy-plane.

34. Average height of a cone Find the average height of the (single) cone $z = \sqrt{x^2 + y^2}$ above the disk $x^2 + y^2 \leq a^2$ in the xy-plane.

35. Average distance from interior of disk to center Find the average distance from a point $P(x, y)$ in the disk $x^2 + y^2 \leq a^2$ to the origin.

36. Average distance squared from a point in a disk to a point in its boundary Find the average value of the *square* of the distance from the point $P(x, y)$ in the disk $x^2 + y^2 \leq 1$ to the boundary point $A(1, 0)$.

Theory and Examples

37. Converting to a polar integral Integrate $f(x, y) = [\ln(x^2 + y^2)]/\sqrt{x^2 + y^2}$ over the region $1 \leq x^2 + y^2 \leq e$.

38. Converting to a polar integral Integrate $f(x, y) = [\ln(x^2 + y^2)]/(x^2 + y^2)$ over the region $1 \leq x^2 + y^2 \leq e^2$.

39. Volume of noncircular right cylinder The region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.

40. Volume of noncircular right cylinder The region enclosed by the lemniscate $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2 - r^2}$. Find the cylinder's volume.

41. Converting to polar integrals

a. The usual way to evaluate the improper integral $I = \int_0^\infty e^{-x^2}\, dx$ is first to calculate its square:

$$I^2 = \left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-y^2}\, dy\right) = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

Evaluate the last integral using polar coordinates and solve the resulting equation for I.

b. Evaluate

$$\lim_{x\to\infty} \operatorname{erf}(x) = \lim_{x\to\infty} \int_0^x \frac{2e^{-t^2}}{\sqrt{\pi}}\, dt.$$

42. Converting to a polar integral Evaluate the integral

$$\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2}\, dx\, dy.$$

43. Existence Integrate the function $f(x, y) = 1/(1 - x^2 - y^2)$ over the disk $x^2 + y^2 \le 3/4$. Does the integral of $f(x, y)$ over the disk $x^2 + y^2 \le 1$ exist? Give reasons for your answer.

44. Area formula in polar coordinates Use the double integral in polar coordinates to derive the formula

$$A = \int_\alpha^\beta \frac{1}{2} r^2\, d\theta$$

for the area of the fan-shaped region between the origin and polar curve $r = f(\theta)$, $\alpha \le \theta \le \beta$.

45. Average distance to a given point inside a disk Let P_0 be a point inside a circle of radius a and let h denote the distance from P_0 to the center of the circle. Let d denote the distance from an arbitrary point P to P_0. Find the average value of d^2 over the region enclosed by the circle. (*Hint:* Simplify your work by placing the center of the circle at the origin and P_0 on the x-axis.)

46. Area Suppose that the area of a region in the polar coordinate plane is

$$A = \int_{\pi/4}^{3\pi/4} \int_{\csc\theta}^{2\sin\theta} r\, dr\, d\theta.$$

Sketch the region and find its area.

COMPUTER EXPLORATIONS

In Exercises 47–50, use a CAS to change the Cartesian integrals into an equivalent polar integral and evaluate the polar integral. Perform the following steps in each exercise.

a. Plot the Cartesian region of integration in the xy-plane.

b. Change each boundary curve of the Cartesian region in part (a) to its polar representation by solving its Cartesian equation for r and θ.

c. Using the results in part (b), plot the polar region of integration in the $r\theta$-plane.

d. Change the integrand from Cartesian to polar coordinates. Determine the limits of integration from your plot in part (c) and evaluate the polar integral using the CAS integration utility.

47. $\displaystyle\int_0^1 \int_x^1 \frac{y}{x^2 + y^2}\, dy\, dx$

48. $\displaystyle\int_0^1 \int_0^{x/2} \frac{x}{x^2 + y^2}\, dy\, dx$

49. $\displaystyle\int_0^1 \int_{-y/3}^{y/3} \frac{y}{\sqrt{x^2 + y^2}}\, dx\, dy$

50. $\displaystyle\int_0^1 \int_y^{2-y} \sqrt{x + y}\, dx\, dy$

15.5 Triple Integrals in Rectangular Coordinates

Just as double integrals allow us to deal with more general situations than could be handled by single integrals, triple integrals enable us to solve still more general problems. We use triple integrals to calculate the volumes of three-dimensional shapes and the average value of a function over a three-dimensional region. Triple integrals also arise in the study of vector fields and fluid flow in three dimensions, as we will see in Chapter 16.

Triple Integrals

If $F(x, y, z)$ is a function defined on a closed, bounded region D in space, such as the region occupied by a solid ball or a lump of clay, then the integral of F over D may be defined in the following way. We partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axes (Figure 15.29). We number the cells that lie completely inside D from 1 to n in some order, the kth cell having dimensions Δx_k by Δy_k by Δz_k and volume $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$. We choose a point (x_k, y_k, z_k) in each cell and form the sum

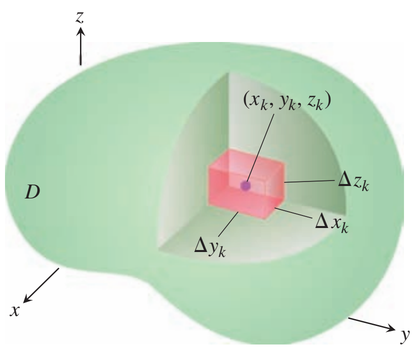


FIGURE 15.29 Partitioning a solid with rectangular cells of volume ΔV_k.

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\, \Delta V_k. \tag{1}$$

We are interested in what happens as D is partitioned by smaller and smaller cells, so that Δx_k, Δy_k, Δz_k and the norm of the partition $\|P\|$, the largest value among Δx_k, Δy_k, Δz_k, all approach zero. When a single limiting value is attained, no matter how the partitions and points (x_k, y_k, z_k) are chosen, we say that F is **integrable** over D. As before, it can be

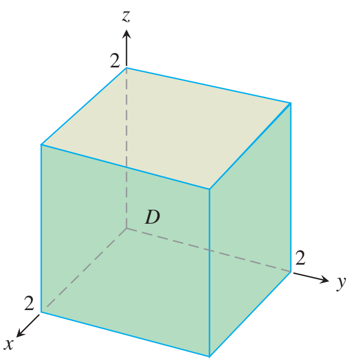


FIGURE 15.33 The region of integration in Example 4.

EXAMPLE 4 Find the average value of $F(x, y, z) = xyz$ throughout the cubical region D bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$ in the first octant.

Solution We sketch the cube with enough detail to show the limits of integration (Figure 15.33). We then use Equation (2) to calculate the average value of F over the cube.

The volume of the region D is $(2)(2)(2) = 8$. The value of the integral of F over the cube is

$$\int_0^2\int_0^2\int_0^2 xyz\,dx\,dy\,dz = \int_0^2\int_0^2 \left[\frac{x^2}{2}yz\right]_{x=0}^{x=2} dy\,dz = \int_0^2\int_0^2 2yz\,dy\,dz$$

$$= \int_0^2 \left[y^2 z\right]_{y=0}^{y=2} dz = \int_0^2 4z\,dz = \left[2z^2\right]_0^2 = 8.$$

With these values, Equation (2) gives

$$\begin{array}{c}\text{Average value of}\\ xyz \text{ over the cube}\end{array} = \frac{1}{\text{volume}}\iiint_{\text{cube}} xyz\,dV = \left(\frac{1}{8}\right)(8) = 1.$$

In evaluating the integral, we chose the order $dx\,dy\,dz$, but any of the other five possible orders would have done as well. ■

Properties of Triple Integrals

Triple integrals have the same algebraic properties as double and single integrals. Simply replace the double integrals in the four properties given in Section 15.2, page 864, with triple integrals.

Exercises 15.5

Triple Integrals in Different Iteration Orders

1. Evaluate the integral in Example 2 taking $F(x, y, z) = 1$ to find the volume of the tetrahedron in the order $dz\,dx\,dy$.

2. **Volume of rectangular solid** Write six different iterated triple integrals for the volume of the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 2$, and $z = 3$. Evaluate one of the integrals.

3. **Volume of tetrahedron** Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.

4. **Volume of solid** Write six different iterated triple integrals for the volume of the region in the first octant enclosed by the cylinder $x^2 + z^2 = 4$ and the plane $y = 3$. Evaluate one of the integrals.

5. **Volume enclosed by paraboloids** Let D be the region bounded by the paraboloids $z = 8 - x^2 - y^2$ and $z = x^2 + y^2$. Write six different triple iterated integrals for the volume of D. Evaluate one of the integrals.

6. **Volume inside paraboloid beneath a plane** Let D be the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$. Write triple iterated integrals in the order $dz\,dx\,dy$ and $dz\,dy\,dx$ that give the volume of D. Do not evaluate either integral.

Evaluating Triple Iterated Integrals

Evaluate the integrals in Exercises 7–20.

7. $\displaystyle\int_0^1\int_0^1\int_0^1 (x^2 + y^2 + z^2)\,dz\,dy\,dx$

8. $\displaystyle\int_0^{\sqrt{2}}\int_0^{3y}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy$

9. $\displaystyle\int_1^e\int_1^{e^2}\int_1^{e^3} \frac{1}{xyz}\,dx\,dy\,dz$

10. $\displaystyle\int_0^1\int_0^{3-3x}\int_0^{3-3x-y} dz\,dy\,dx$

11. $\displaystyle\int_0^{\pi/6}\int_0^1\int_{-2}^3 y\sin z\,dx\,dy\,dz$

12. $\displaystyle\int_{-1}^1\int_0^1\int_0^2 (x + y + z)\,dy\,dx\,dz$

13. $\displaystyle\int_0^3\int_0^{\sqrt{9-x^2}}\int_0^{\sqrt{9-x^2}} dz\,dy\,dx$

14. $\displaystyle\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_0^{2x+y} dz\,dx\,dy$

15. $\displaystyle\int_0^1\int_0^{2-x}\int_0^{2-x-y} dz\,dy\,dx$

16. $\displaystyle\int_0^1\int_0^{1-x^2}\int_3^{4-x^2-y} x\,dz\,dy\,dx$

17. $\displaystyle\int_0^\pi\int_0^\pi\int_0^\pi \cos(u + v + w)\,du\,dv\,dw$ (uvw-space)

18. $\displaystyle\int_0^1\int_1^{\sqrt{e}}\int_1^e se^s \ln r\,\frac{(\ln t)^2}{t}\,dt\,dr\,ds$ (rst-space)

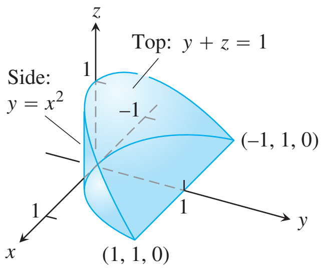
19. $\displaystyle\int_0^{\pi/4}\int_0^{\ln \sec v}\int_{-\infty}^{2t} e^x \, dx \, dt \, dv \quad (tvx\text{-space})$

20. $\displaystyle\int_0^7\int_0^2\int_0^{\sqrt{4-q^2}} \frac{q}{r+1} \, dp \, dq \, dr \quad (pqr\text{-space})$

Finding Equivalent Iterated Integrals

21. Here is the region of integration of the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_0^{1-y} dz \, dy \, dx.$$



Rewrite the integral as an equivalent iterated integral in the order

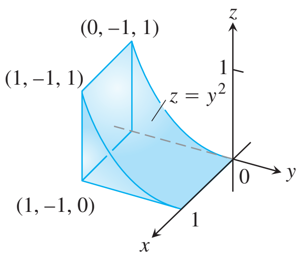
a. $dy \, dz \, dx$ **b.** $dy \, dx \, dz$

c. $dx \, dy \, dz$ **d.** $dx \, dz \, dy$

e. $dz \, dx \, dy$.

22. Here is the region of integration of the integral

$$\int_0^1\int_{-1}^{0}\int_0^{y^2} dz \, dy \, dx.$$



Rewrite the integral as an equivalent iterated integral in the order

a. $dy \, dz \, dx$ **b.** $dy \, dx \, dz$

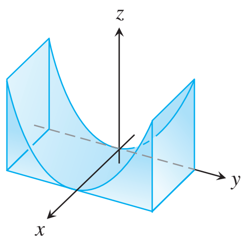
c. $dx \, dy \, dz$ **d.** $dx \, dz \, dy$

e. $dz \, dx \, dy$.

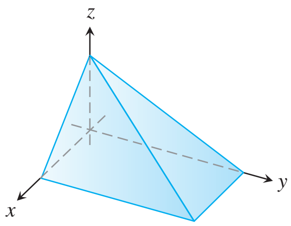
Finding Volumes Using Triple Integrals

Find the volumes of the regions in Exercises 23–36.

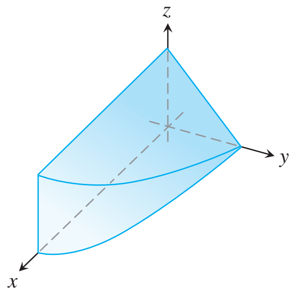
23. The region between the cylinder $z = y^2$ and the xy-plane that is bounded by the planes $x = 0, x = 1, y = -1, y = 1$



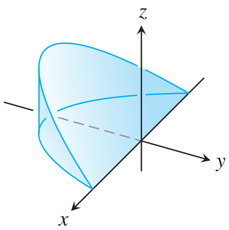
24. The region in the first octant bounded by the coordinate planes and the planes $x + z = 1, y + 2z = 2$



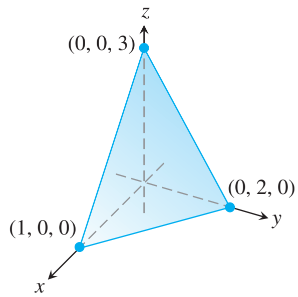
25. The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$



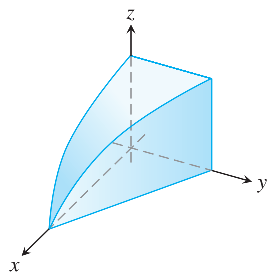
26. The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$



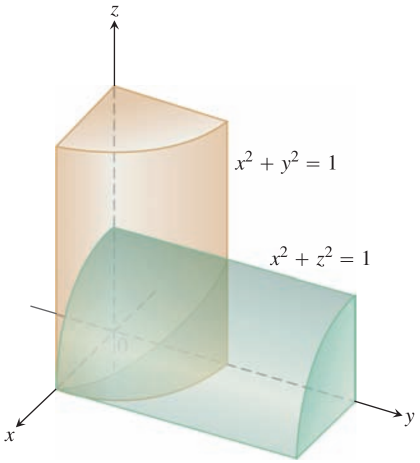
27. The tetrahedron in the first octant bounded by the coordinate planes and the plane passing through (1, 0, 0), (0, 2, 0), and (0, 0, 3)



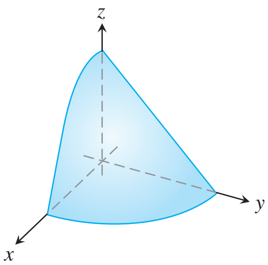
28. The region in the first octant bounded by the coordinate planes, the plane $y = 1 - x$, and the surface $z = \cos(\pi x/2)$, $0 \le x \le 1$



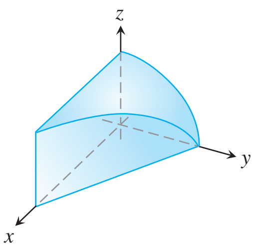
29. The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure



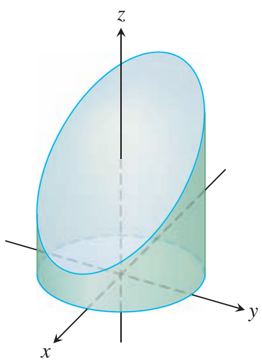
30. The region in the first octant bounded by the coordinate planes and the surface $z = 4 - x^2 - y$



31. The region in the first octant bounded by the coordinate planes, the plane $x + y = 4$, and the cylinder $y^2 + 4z^2 = 16$



32. The region cut from the cylinder $x^2 + y^2 = 4$ by the plane $z = 0$ and the plane $x + z = 3$



33. The region between the planes $x + y + 2z = 2$ and $2x + 2y + z = 4$ in the first octant

34. The finite region bounded by the planes $z = x$, $x + z = 8$, $z = y$, $y = 8$, and $z = 0$

35. The region cut from the solid elliptical cylinder $x^2 + 4y^2 \le 4$ by the xy-plane and the plane $z = x + 2$

36. The region bounded in back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the xy-plane

Average Values

In Exercises 37–40, find the average value of $F(x, y, z)$ over the given region.

37. $F(x, y, z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

38. $F(x, y, z) = x + y - z$ over the rectangular solid in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 2$

39. $F(x, y, z) = x^2 + y^2 + z^2$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 1$, $y = 1$, and $z = 1$

40. $F(x, y, z) = xyz$ over the cube in the first octant bounded by the coordinate planes and the planes $x = 2$, $y = 2$, and $z = 2$

Changing the Order of Integration

Evaluate the integrals in Exercises 41–44 by changing the order of integration in an appropriate way.

41. $\displaystyle\int_0^4\int_0^1\int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$

42. $\displaystyle\int_0^1\int_0^1\int_{x^2}^1 12xze^{zy^2}\,dy\,dx\,dz$

43. $\displaystyle\int_0^1\int_{\sqrt[3]{z}}^1\int_0^{\ln 3} \frac{\pi e^{2x}\sin \pi y^2}{y^2}\,dx\,dy\,dz$

44. $\displaystyle\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx$

Theory and Examples

45. Finding an upper limit of an iterated integral Solve for a:

$$\int_0^1\int_0^{4-a-x^2}\int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15}.$$

46. Ellipsoid For what value of c is the volume of the ellipsoid $x^2 + (y/2)^2 + (z/c)^2 = 1$ equal to 8π?

47. Minimizing a triple integral What domain D in space minimizes the value of the integral

$$\iiint_D (4x^2 + 4y^2 + z^2 - 4)\,dV\,?$$

Give reasons for your answer.

48. Maximizing a triple integral What domain D in space maximizes the value of the integral

$$\iiint_D (1 - x^2 - y^2 - z^2)\,dV\,?$$

Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 49–52, use a CAS integration utility to evaluate the triple integral of the given function over the specified solid region.

49. $F(x, y, z) = x^2y^2z$ over the solid cylinder bounded by $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 1$

50. $F(x, y, z) = |xyz|$ over the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$

51. $F(x, y, z) = \dfrac{z}{(x^2 + y^2 + z^2)^{3/2}}$ over the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$

52. $F(x, y, z) = x^4 + y^2 + z^2$ over the solid sphere $x^2 + y^2 + z^2 \le 1$

15.6 Moments and Centers of Mass

This section shows how to calculate the masses and moments of two- and three-dimensional objects in Cartesian coordinates. Section 15.7 gives the calculations for cylindrical and spherical coordinates. The definitions and ideas are similar to the single-variable case we studied in Section 6.6, but now we can consider more realistic situations.

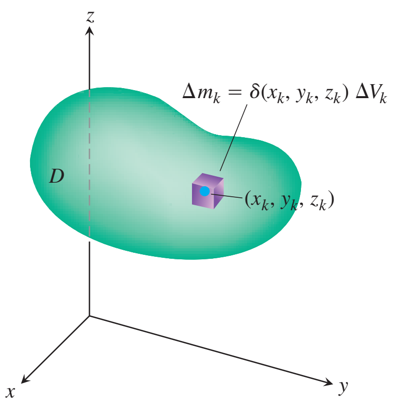


FIGURE 15.34 To define an object's mass, we first imagine it to be partitioned into a finite number of mass elements Δm_k.

Masses and First Moments

If $\delta(x, y, z)$ is the density (mass per unit volume) of an object occupying a region D in space, the integral of δ over D gives the **mass** of the object. To see why, imagine partitioning the object into n mass elements like the one in Figure 15.34. The object's mass is the limit

$$M = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta m_k = \lim_{n\to\infty} \sum_{k=1}^{n} \delta(x_k, y_k, z_k)\,\Delta V_k = \iiint_D \delta(x, y, z)\,dV.$$

The *first moment* of a solid region D about a coordinate plane is defined as the triple integral over D of the distance from a point (x, y, z) in D to the plane multiplied by the density of the solid at that point. For instance, the first moment about the yz-plane is the integral

$$M_{yz} = \iiint_D x\delta(x, y, z)\,dV.$$

The *center of mass* is found from the first moments. For instance, the x-coordinate of the center of mass is $\bar{x} = M_{yz}/M$.

For a two-dimensional object, such as a thin, flat plate, we calculate first moments about the coordinate axes by simply dropping the z-coordinate. So the first moment about the y-axis is the double integral over the region R forming the plate of the distance from the axis multiplied by the density, or

$$M_y = \iint_R x\delta(x, y)\,dA.$$

Table 15.1 summarizes the formulas.

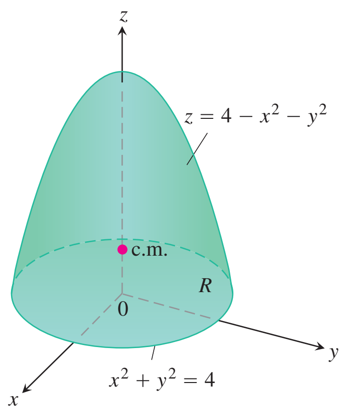


FIGURE 15.35 Finding the center of mass of a solid (Example 1).

EXAMPLE 1 Find the center of mass of a solid of constant density δ bounded below by the disk $R\colon x^2 + y^2 \le 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.35).

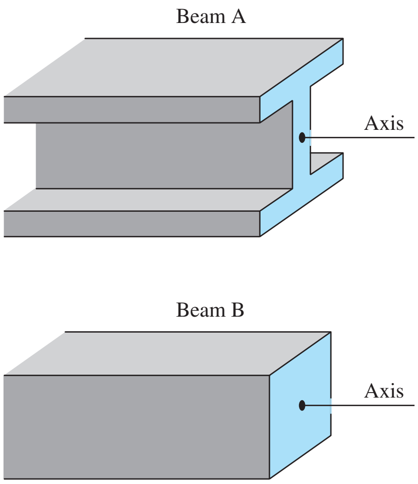


FIGURE 15.41 The greater the polar moment of inertia of the cross-section of a beam about the beam's longitudinal axis, the stiffer the beam. Beams A and B have the same cross-sectional area, but A is stiffer.

Similarly, the moment of inertia about the y-axis is

$$I_y = \int_0^1 \int_0^{2x} x^2 \delta(x, y)\, dy\, dx = \frac{39}{5}.$$

Notice that we integrate y^2 times density in calculating I_x and x^2 times density to find I_y.

Since we know I_x and I_y, we do not need to evaluate an integral to find I_0; we can use the equation $I_0 = I_x + I_y$ from Table 15.2 instead:

$$I_0 = 12 + \frac{39}{5} = \frac{60 + 39}{5} = \frac{99}{5}.$$

The moment of inertia also plays a role in determining how much a horizontal metal beam will bend under a load. The stiffness of the beam is a constant times I, the moment of inertia of a typical cross-section of the beam about the beam's longitudinal axis. The greater the value of I, the stiffer the beam and the less it will bend under a given load. That is why we use I-beams instead of beams whose cross-sections are square. The flanges at the top and bottom of the beam hold most of the beam's mass away from the longitudinal axis to increase the value of I (Figure 15.41).

Exercises 15.6

Plates of Constant Density

1. **Finding a center of mass** Find the center of mass of a thin plate of density $\delta = 3$ bounded by the lines $x = 0$, $y = x$, and the parabola $y = 2 - x^2$ in the first quadrant.

2. **Finding moments of inertia** Find the moments of inertia about the coordinate axes of a thin rectangular plate of constant density δ bounded by the lines $x = 3$ and $y = 3$ in the first quadrant.

3. **Finding a centroid** Find the centroid of the region in the first quadrant bounded by the x-axis, the parabola $y^2 = 2x$, and the line $x + y = 4$.

4. **Finding a centroid** Find the centroid of the triangular region cut from the first quadrant by the line $x + y = 3$.

5. **Finding a centroid** Find the centroid of the region cut from the first quadrant by the circle $x^2 + y^2 = a^2$.

6. **Finding a centroid** Find the centroid of the region between the x-axis and the arch $y = \sin x$, $0 \le x \le \pi$.

7. **Finding moments of inertia** Find the moment of inertia about the x-axis of a thin plate of density $\delta = 1$ bounded by the circle $x^2 + y^2 = 4$. Then use your result to find I_y and I_0 for the plate.

8. **Finding a moment of inertia** Find the moment of inertia with respect to the y-axis of a thin sheet of constant density $\delta = 1$ bounded by the curve $y = (\sin^2 x)/x^2$ and the interval $\pi \le x \le 2\pi$ of the x-axis.

9. **The centroid of an infinite region** Find the centroid of the infinite region in the second quadrant enclosed by the coordinate axes and the curve $y = e^x$. (Use improper integrals in the mass-moment formulas.)

10. **The first moment of an infinite plate** Find the first moment about the y-axis of a thin plate of density $\delta(x, y) = 1$ covering the infinite region under the curve $y = e^{-x^2/2}$ in the first quadrant.

Plates with Varying Density

11. **Finding a moment of inertia** Find the moment of inertia about the x-axis of a thin plate bounded by the parabola $x = y - y^2$ and the line $x + y = 0$ if $\delta(x, y) = x + y$.

12. **Finding mass** Find the mass of a thin plate occupying the smaller region cut from the ellipse $x^2 + 4y^2 = 12$ by the parabola $x = 4y^2$ if $\delta(x, y) = 5x$.

13. **Finding a center of mass** Find the center of mass of a thin triangular plate bounded by the y-axis and the lines $y = x$ and $y = 2 - x$ if $\delta(x, y) = 6x + 3y + 3$.

14. **Finding a center of mass and moment of inertia** Find the center of mass and moment of inertia about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$.

15. **Center of mass, moment of inertia** Find the center of mass and the moment of inertia about the y-axis of a thin rectangular plate cut from the first quadrant by the lines $x = 6$ and $y = 1$ if $\delta(x, y) = x + y + 1$.

16. **Center of mass, moment of inertia** Find the center of mass and the moment of inertia about the y-axis of a thin plate bounded by the line $y = 1$ and the parabola $y = x^2$ if the density is $\delta(x, y) = y + 1$.

17. **Center of mass, moment of inertia** Find the center of mass and the moment of inertia about the y-axis of a thin plate bounded by the x-axis, the lines $x = \pm 1$, and the parabola $y = x^2$ if $\delta(x, y) = 7y + 1$.

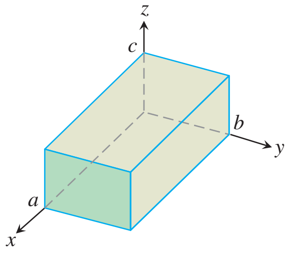
18. Center of mass, moment of inertia Find the center of mass and the moment of inertia about the x-axis of a thin rectangular plate bounded by the lines $x = 0$, $x = 20$, $y = -1$, and $y = 1$ if $\delta(x, y) = 1 + (x/20)$.

19. Center of mass, moments of inertia Find the center of mass, the moment of inertia about the coordinate axes, and the polar moment of inertia of a thin triangular plate bounded by the lines $y = x$, $y = -x$, and $y = 1$ if $\delta(x, y) = y + 1$.

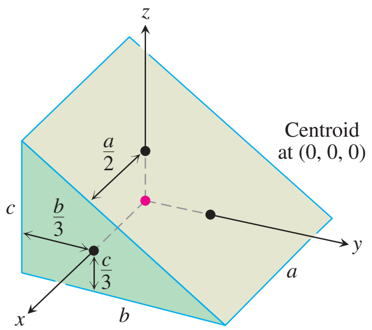
20. Center of mass, moments of inertia Repeat Exercise 19 for $\delta(x, y) = 3x^2 + 1$.

Solids with Constant Density

21. Moments of inertia Find the moments of inertia of the rectangular solid shown here with respect to its edges by calculating I_x, I_y, and I_z.



22. Moments of inertia The coordinate axes in the figure run through the centroid of a solid wedge parallel to the labeled edges. Find I_x, I_y, and I_z if $a = b = 6$ and $c = 4$.



23. Center of mass and moments of inertia A solid "trough" of constant density is bounded below by the surface $z = 4y^2$, above by the plane $z = 4$, and on the ends by the planes $x = 1$ and $x = -1$. Find the center of mass and the moments of inertia with respect to the three axes.

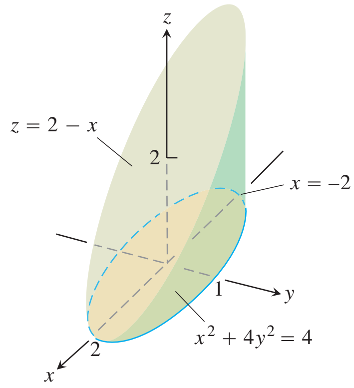
24. Center of mass A solid of constant density is bounded below by the plane $z = 0$, on the sides by the elliptical cylinder $x^2 + 4y^2 = 4$, and above by the plane $z = 2 - x$ (see the accompanying figure).

a. Find $\bar{x}$ and $\bar{y}$.

b. Evaluate the integral

$$M_{xy} = \int_{-2}^{2}\int_{-(1/2)\sqrt{4-x^2}}^{(1/2)\sqrt{4-x^2}}\int_{0}^{2-x} z\,dz\,dy\,dx$$

using integral tables to carry out the final integration with respect to x. Then divide M_{xy} by M to verify that $\bar{z} = 5/4$.



25. a. Center of mass Find the center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$.

b. Find the plane $z = c$ that divides the solid into two parts of equal volume. This plane does not pass through the center of mass.

26. Moments A solid cube, 2 units on a side, is bounded by the planes $x = \pm 1$, $z = \pm 1$, $y = 3$, and $y = 5$. Find the center of mass and the moments of inertia about the coordinate axes.

27. Moment of inertia about a line A wedge like the one in Exercise 22 has $a = 4$, $b = 6$, and $c = 3$. Make a quick sketch to check for yourself that the square of the distance from a typical point (x, y, z) of the wedge to the line $L: z = 0, y = 6$ is $r^2 = (y - 6)^2 + z^2$. Then calculate the moment of inertia of the wedge about L.

28. Moment of inertia about a line A wedge like the one in Exercise 22 has $a = 4$, $b = 6$, and $c = 3$. Make a quick sketch to check for yourself that the square of the distance from a typical point (x, y, z) of the wedge to the line $L: x = 4, y = 0$ is $r^2 = (x - 4)^2 + y^2$. Then calculate the moment of inertia of the wedge about L.

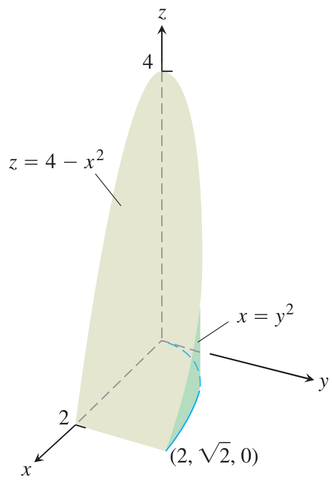
Solids with Varying Density

In Exercises 29 and 30, find

a. the mass of the solid. **b.** the center of mass.

29. A solid region in the first octant is bounded by the coordinate planes and the plane $x + y + z = 2$. The density of the solid is $\delta(x, y, z) = 2x$.

30. A solid in the first octant is bounded by the planes $y = 0$ and $z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ (see the accompanying figure). Its density function is $\delta(x, y, z) = kxy$, k a constant.



In Exercises 31 and 32, find

a. the mass of the solid. **b.** the center of mass.

c. the moments of inertia about the coordinate axes.

31. A solid cube in the first octant is bounded by the coordinate planes and by the planes $x = 1$, $y = 1$, and $z = 1$. The density of the cube is $\delta(x, y, z) = x + y + z + 1$.

32. A wedge like the one in Exercise 22 has dimensions $a = 2$, $b = 6$, and $c = 3$. The density is $\delta(x, y, z) = x + 1$. Notice that if the density is constant, the center of mass will be (0, 0, 0).

33. Mass Find the mass of the solid bounded by the planes $x + z = 1$, $x - z = -1$, $y = 0$ and the surface $y = \sqrt{z}$. The density of the solid is $\delta(x, y, z) = 2y + 5$.

34. Mass Find the mass of the solid region bounded by the parabolic surfaces $z = 16 - 2x^2 - 2y^2$ and $z = 2x^2 + 2y^2$ if the density of the solid is $\delta(x, y, z) = \sqrt{x^2 + y^2}$.

Theory and Examples

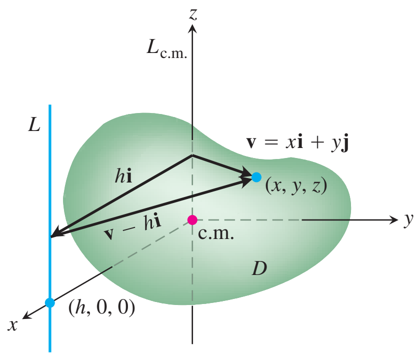
The Parallel Axis Theorem Let $L_{\text{c.m.}}$ be a line through the center of mass of a body of mass m and let L be a parallel line h units away from $L_{\text{c.m.}}$. The *Parallel Axis Theorem* says that the moments of inertia $I_{\text{c.m.}}$ and I_L of the body about $L_{\text{c.m.}}$ and L satisfy the equation

$$I_L = I_{\text{c.m.}} + mh^2. \tag{2}$$

As in the two-dimensional case, the theorem gives a quick way to calculate one moment when the other moment and the mass are known.

35. Proof of the Parallel Axis Theorem

a. Show that the first moment of a body in space about any plane through the body's center of mass is zero. (*Hint:* Place the body's center of mass at the origin and let the plane be the yz-plane. What does the formula $\bar{x} = M_{yz}/M$ then tell you?)



b. To prove the Parallel Axis Theorem, place the body with its center of mass at the origin, with the line $L_{\text{c.m.}}$ along the z-axis and the line L perpendicular to the xy-plane at the point $(h, 0, 0)$. Let D be the region of space occupied by the body. Then, in the notation of the figure,

$$I_L = \iiint_D |\mathbf{v} - h\mathbf{i}|^2 \, dm.$$

Expand the integrand in this integral and complete the proof.

36. The moment of inertia about a diameter of a solid sphere of constant density and radius a is $(2/5)ma^2$, where m is the mass of the sphere. Find the moment of inertia about a line tangent to the sphere.

37. The moment of inertia of the solid in Exercise 21 about the z-axis is $I_z = abc(a^2 + b^2)/3$.

a. Use Equation (2) to find the moment of inertia of the solid about the line parallel to the z-axis through the solid's center of mass.

b. Use Equation (2) and the result in part (a) to find the moment of inertia of the solid about the line $x = 0$, $y = 2b$.

38. If $a = b = 6$ and $c = 4$, the moment of inertia of the solid wedge in Exercise 22 about the x-axis is $I_x = 208$. Find the moment of inertia of the wedge about the line $y = 4$, $z = -4/3$ (the edge of the wedge's narrow end).

15.7 Triple Integrals in Cylindrical and Spherical Coordinates

When a calculation in physics, engineering, or geometry involves a cylinder, cone, or sphere, we can often simplify our work by using cylindrical or spherical coordinates, which are introduced in this section. The procedure for transforming to these coordinates and evaluating the resulting triple integrals is similar to the transformation to polar coordinates in the plane studied in Section 15.4.

Integration in Cylindrical Coordinates

We obtain cylindrical coordinates for space by combining polar coordinates in the xy-plane with the usual z-axis. This assigns to every point in space one or more coordinate triples of the form (r, θ, z), as shown in Figure 15.42.

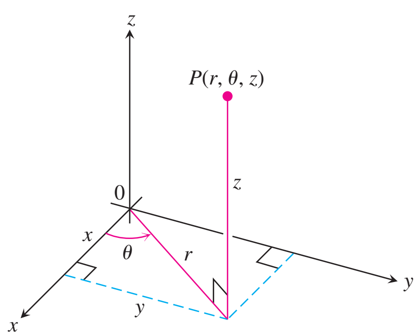


FIGURE 15.42 The cylindrical coordinates of a point in space are r, θ, and z.

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

In the next section we offer a more general procedure for determining dV in cylindrical and spherical coordinates. The results, of course, will be the same.

Exercises 15.7

Evaluating Integrals in Cylindrical Coordinates

Evaluate the cylindrical coordinate integrals in Exercises 1–6.

1. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{\sqrt{2-r^2}} dz\,r\,dr\,d\theta$
2. $\displaystyle\int_0^{2\pi}\int_0^3\int_{r^2/3}^{\sqrt{18-r^2}} dz\,r\,dr\,d\theta$
3. $\displaystyle\int_0^{2\pi}\int_0^{\theta/2\pi}\int_0^{3+24r^2} dz\,r\,dr\,d\theta$
4. $\displaystyle\int_0^{\pi}\int_0^{\theta/\pi}\int_{-\sqrt{4-r^2}}^{3\sqrt{4-r^2}} z\,dz\,r\,dr\,d\theta$
5. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^{1/\sqrt{2-r^2}} 3\,dz\,r\,dr\,d\theta$
6. $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}(r^2\sin^2\theta + z^2)\,dz\,r\,dr\,d\theta$

Changing the Order of Integration in Cylindrical Coordinates

The integrals we have seen so far suggest that there are preferred orders of integration for cylindrical coordinates, but other orders usually work well and are occasionally easier to evaluate. Evaluate the integrals in Exercises 7–10.

7. $\displaystyle\int_0^{2\pi}\int_0^3\int_0^{z/3} r^3\,dr\,dz\,d\theta$
8. $\displaystyle\int_{-1}^1\int_0^{2\pi}\int_0^{1+\cos\theta} 4r\,dr\,d\theta\,dz$
9. $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}(r^2\cos^2\theta + z^2)\,r\,d\theta\,dr\,dz$
10. $\displaystyle\int_0^2\int_{r-2}^{\sqrt{4-r^2}}\int_0^{2\pi}(r\sin\theta + 1)\,r\,d\theta\,dz\,dr$
11. Let D be the region bounded below by the plane $z = 0$, above by the sphere $x^2 + y^2 + z^2 = 4$, and on the sides by the cylinder $x^2 + y^2 = 1$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

    a. $dz\,dr\,d\theta$ **b.** $dr\,dz\,d\theta$ **c.** $d\theta\,dz\,dr$

12. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$. Set up the triple integrals in cylindrical coordinates that give the volume of D using the following orders of integration.

    a. $dz\,dr\,d\theta$ **b.** $dr\,dz\,d\theta$ **c.** $d\theta\,dz\,dr$

Finding Iterated Integrals in Cylindrical Coordinates

13. Give the limits of integration for evaluating the integral

$$\iiint f(r, \theta, z)\,dz\,r\,dr\,d\theta$$

as an iterated integral over the region that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos\theta$, and on top by the paraboloid $z = 3r^2$.

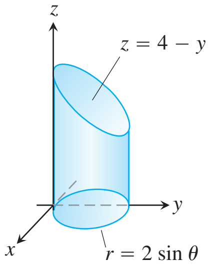
14. Convert the integral

$$\int_{-1}^1\int_0^{\sqrt{1-y^2}}\int_0^x (x^2 + y^2)\,dz\,dx\,dy$$

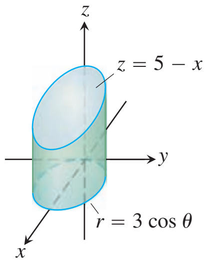
to an equivalent integral in cylindrical coordinates and evaluate the result.

In Exercises 15–20, set up the iterated integral for evaluating $\iiint_D f(r, \theta, z)\,dz\,r\,dr\,d\theta$ over the given region D.

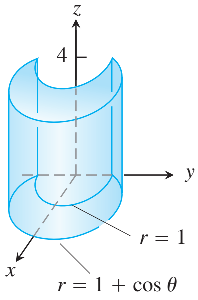
15. D is the right circular cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$.



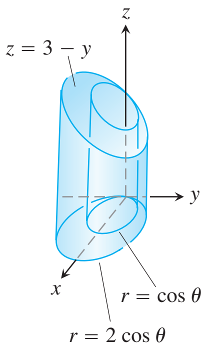
16. D is the right circular cylinder whose base is the circle $r = 3 \cos \theta$ and whose top lies in the plane $z = 5 - x$.



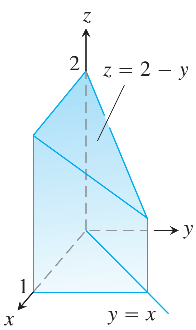
17. D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos \theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



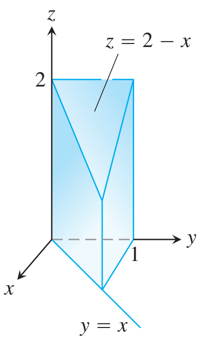
18. D is the solid right cylinder whose base is the region between the circles $r = \cos \theta$ and $r = 2 \cos \theta$ and whose top lies in the plane $z = 3 - y$.



19. D is the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = 2 - y$.



20. D is the prism whose base is the triangle in the xy-plane bounded by the y-axis and the lines $y = x$ and $y = 1$ and whose top lies in the plane $z = 2 - x$.



Evaluating Integrals in Spherical Coordinates

Evaluate the spherical coordinate integrals in Exercises 21–26.

21. $\displaystyle\int_0^{\pi}\int_0^{\pi}\int_0^{2\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

22. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} (\rho\cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

23. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

24. $\displaystyle\int_0^{3\pi/2}\int_0^{\pi}\int_0^{1} 5\rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$

25. $\displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} 3\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

26. $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} (\rho\cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Changing the Order of Integration in Spherical Coordinates

The previous integrals suggest there are preferred orders of integration for spherical coordinates, but other orders give the same value and are occasionally easier to evaluate. Evaluate the integrals in Exercises 27–30.

27. $\displaystyle\int_0^{2}\int_{-\pi}^{0}\int_{\pi/4}^{\pi/2} \rho^3 \sin 2\phi \, d\phi \, d\theta \, d\rho$

28. $\displaystyle\int_{\pi/6}^{\pi/3}\int_{\csc\phi}^{2\csc\phi}\int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\rho \, d\phi$

29. $\displaystyle\int_0^{1}\int_0^{\pi}\int_0^{\pi/4} 12\rho \sin^3\phi \, d\phi \, d\theta \, d\rho$

30. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2} 5\rho^4 \sin^3\phi \, d\rho \, d\theta \, d\phi$

31. Let D be the region in Exercise 11. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

a. $d\rho \, d\phi \, d\theta$ **b.** $d\phi \, d\rho \, d\theta$

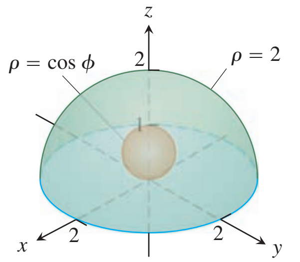
32. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration.

a. $d\rho \, d\phi \, d\theta$ **b.** $d\phi \, d\rho \, d\theta$

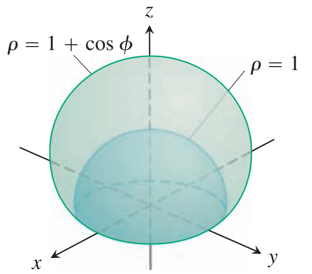
Finding Iterated Integrals in Spherical Coordinates

In Exercises 33–38, **(a)** find the spherical coordinate limits for the integral that calculates the volume of the given solid and then **(b)** evaluate the integral.

33. The solid between the sphere $\rho = \cos\phi$ and the hemisphere $\rho = 2, z \geq 0$



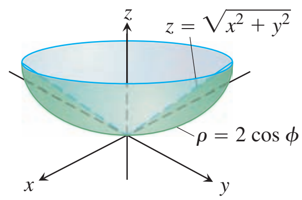
34. The solid bounded below by the hemisphere $\rho = 1, z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos\phi$



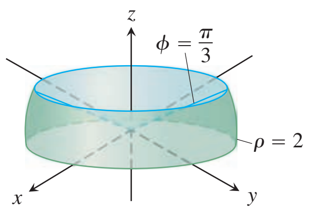
35. The solid enclosed by the cardioid of revolution $\rho = 1 - \cos\phi$

36. The upper portion cut from the solid in Exercise 35 by the xy-plane

37. The solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2 + y^2}$



38. The solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/3$



Finding Triple Integrals

39. Set up triple integrals for the volume of the sphere $\rho = 2$ in **(a)** spherical, **(b)** cylindrical, and **(c)** rectangular coordinates.

40. Let D be the region in the first octant that is bounded below by the cone $\phi = \pi/4$ and above by the sphere $\rho = 3$. Express the volume of D as an iterated triple integral in **(a)** cylindrical and **(b)** spherical coordinates. Then **(c)** find V.

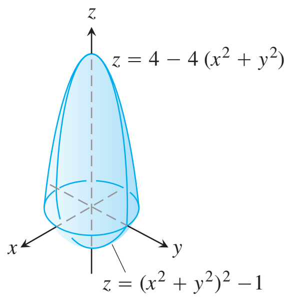
41. Let D be the smaller cap cut from a solid ball of radius 2 units by a plane 1 unit from the center of the sphere. Express the volume of D as an iterated triple integral in **(a)** spherical, **(b)** cylindrical, and **(c)** rectangular coordinates. Then **(d)** find the volume by evaluating one of the three triple integrals.

42. Express the moment of inertia I_z of the solid hemisphere $x^2 + y^2 + z^2 \leq 1, z \geq 0$, as an iterated integral in **(a)** cylindrical and **(b)** spherical coordinates. Then **(c)** find I_z.

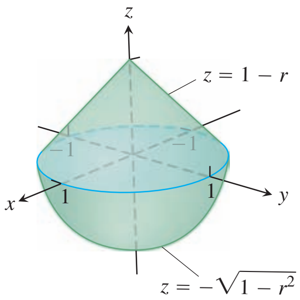
Volumes

Find the volumes of the solids in Exercises 43–48.

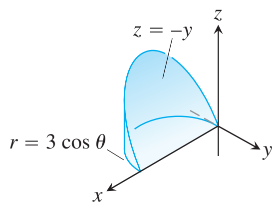
43.



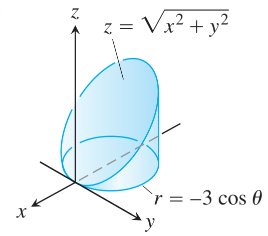
44.



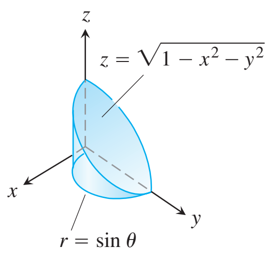
45.



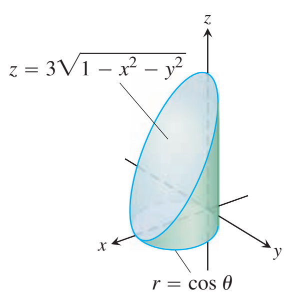
46.



47.



48.



49. Sphere and cones Find the volume of the portion of the solid sphere $\rho \leq a$ that lies between the cones $\phi = \pi/3$ and $\phi = 2\pi/3$.

50. Sphere and half-planes Find the volume of the region cut from the solid sphere $\rho \leq a$ by the half-planes $\theta = 0$ and $\theta = \pi/6$ in the first octant.

51. Sphere and plane Find the volume of the smaller region cut from the solid sphere $\rho \leq 2$ by the plane $z = 1$.

52. Cone and planes Find the volume of the solid enclosed by the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$.

53. Cylinder and paraboloid Find the volume of the region bounded below by the plane $z = 0$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2$.

54. **Cylinder and paraboloids** Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.

55. **Cylinder and cones** Find the volume of the solid cut from the thick-walled cylinder $1 \le x^2 + y^2 \le 2$ by the cones $z = \pm\sqrt{x^2 + y^2}$.

56. **Sphere and cylinder** Find the volume of the region that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside the cylinder $x^2 + y^2 = 1$.

57. **Cylinder and planes** Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

58. **Cylinder and planes** Find the volume of the region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $x + y + z = 4$.

59. **Region trapped by paraboloids** Find the volume of the region bounded above by the paraboloid $z = 5 - x^2 - y^2$ and below by the paraboloid $z = 4x^2 + 4y^2$.

60. **Paraboloid and cylinder** Find the volume of the region bounded above by the paraboloid $z = 9 - x^2 - y^2$, below by the xy-plane, and lying *outside* the cylinder $x^2 + y^2 = 1$.

61. **Cylinder and sphere** Find the volume of the region cut from the solid cylinder $x^2 + y^2 \le 1$ by the sphere $x^2 + y^2 + z^2 = 4$.

62. **Sphere and paraboloid** Find the volume of the region bounded above by the sphere $x^2 + y^2 + z^2 = 2$ and below by the paraboloid $z = x^2 + y^2$.

Average Values

63. Find the average value of the function $f(r, \theta, z) = r$ over the region bounded by the cylinder $r = 1$ between the planes $z = -1$ and $z = 1$.

64. Find the average value of the function $f(r, \theta, z) = r$ over the solid ball bounded by the sphere $r^2 + z^2 = 1$. (This is the sphere $x^2 + y^2 + z^2 = 1$.)

65. Find the average value of the function $f(\rho, \phi, \theta) = \rho$ over the solid ball $\rho \le 1$.

66. Find the average value of the function $f(\rho, \phi, \theta) = \rho\cos\phi$ over the solid upper ball $\rho \le 1, 0 \le \phi \le \pi/2$.

Masses, Moments, and Centroids

67. **Center of mass** A solid of constant density is bounded below by the plane $z = 0$, above by the cone $z = r, r \ge 0$, and on the sides by the cylinder $r = 1$. Find the center of mass.

68. **Centroid** Find the centroid of the region in the first octant that is bounded above by the cone $z = \sqrt{x^2 + y^2}$, below by the plane $z = 0$, and on the sides by the cylinder $x^2 + y^2 = 4$ and the planes $x = 0$ and $y = 0$.

69. **Centroid** Find the centroid of the solid in Exercise 38.

70. **Centroid** Find the centroid of the solid bounded above by the sphere $\rho = a$ and below by the cone $\phi = \pi/4$.

71. **Centroid** Find the centroid of the region that is bounded above by the surface $z = \sqrt{r}$, on the sides by the cylinder $r = 4$, and below by the xy-plane.

72. **Centroid** Find the centroid of the region cut from the solid ball $r^2 + z^2 \le 1$ by the half-planes $\theta = -\pi/3, r \ge 0$, and $\theta = \pi/3, r \ge 0$.

73. **Moment of inertia of solid cone** Find the moment of inertia of a right circular cone of base radius 1 and height 1 about an axis through the vertex parallel to the base. (Take $\delta = 1$.)

74. **Moment of inertia of solid sphere** Find the moment of inertia of a solid sphere of radius a about a diameter. (Take $\delta = 1$.)

75. **Moment of inertia of solid cone** Find the moment of inertia of a right circular cone of base radius a and height h about its axis. (*Hint:* Place the cone with its vertex at the origin and its axis along the z-axis.)

76. **Variable density** A solid is bounded on the top by the paraboloid $z = r^2$, on the bottom by the plane $z = 0$, and on the sides by the cylinder $r = 1$. Find the center of mass and the moment of inertia about the z-axis if the density is

    a. $\delta(r, \theta, z) = z$ **b.** $\delta(r, \theta, z) = r$.

77. **Variable density** A solid is bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Find the center of mass and the moment of inertia about the z-axis if the density is

    a. $\delta(r, \theta, z) = z$ **b.** $\delta(r, \theta, z) = z^2$.

78. **Variable density** A solid ball is bounded by the sphere $\rho = a$. Find the moment of inertia about the z-axis if the density is

    a. $\delta(\rho, \phi, \theta) = \rho^2$ **b.** $\delta(\rho, \phi, \theta) = r = \rho\sin\phi$.

79. **Centroid of solid semiellipsoid** Show that the centroid of the solid semiellipsoid of revolution $(r^2/a^2) + (z^2/h^2) \le 1, z \ge 0$, lies on the z-axis three-eighths of the way from the base to the top. The special case $h = a$ gives a solid hemisphere. Thus, the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base to the top.

80. **Centroid of solid cone** Show that the centroid of a solid right circular cone is one-fourth of the way from the base to the vertex. (In general, the centroid of a solid cone or pyramid is one-fourth of the way from the centroid of the base to the vertex.)

81. **Density of center of a planet** A planet is in the shape of a sphere of radius R and total mass M with spherically symmetric density distribution that increases linearly as one approaches its center. What is the density at the center of this planet if the density at its edge (surface) is taken to be zero?

82. **Mass of planet's atmosphere** A spherical planet of radius R has an atmosphere whose density is $\mu = \mu_0 e^{-ch}$, where h is the altitude above the surface of the planet, μ_0 is the density at sea level, and c is a positive constant. Find the mass of the planet's atmosphere.

Theory and Examples

83. **Vertical planes in cylindrical coordinates**

    a. Show that planes perpendicular to the x-axis have equations of the form $r = a\sec\theta$ in cylindrical coordinates.

    b. Show that planes perpendicular to the y-axis have equations of the form $r = b\csc\theta$.

84. (*Continuation of Exercise 83.*) Find an equation of the form $r = f(\theta)$ in cylindrical coordinates for the plane $ax + by = c$, $c \ne 0$.

85. **Symmetry** What symmetry will you find in a surface that has an equation of the form $r = f(z)$ in cylindrical coordinates? Give reasons for your answer.

86. **Symmetry** What symmetry will you find in a surface that has an equation of the form $\rho = f(\phi)$ in spherical coordinates? Give reasons for your answer.

The Jacobian of the transformation, again from Equations (9), is

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

We now have everything we need to apply Equation (7):

$$\begin{aligned}
&\int_0^3 \int_0^4 \int_{x=y/2}^{x=(y/2)+1} \left(\frac{2x-y}{2} + \frac{z}{3} \right) dx\, dy\, dz \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w) \left| J(u, v, w) \right| du\, dv\, dw \\
&= \int_0^1 \int_0^2 \int_0^1 (u + w)(6)\, du\, dv\, dw = 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dv\, dw \\
&= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dv\, dw = 6 \int_0^1 \left[\frac{v}{2} + vw \right]_0^2 dw = 6 \int_0^1 (1 + 2w)\, dw \\
&= 6\left[w + w^2 \right]_0^1 = 6(2) = 12.
\end{aligned}$$

Exercises 15.8

Jacobians and Transformed Regions in the Plane

1. a. Solve the system

$$u = x - y, \qquad v = 2x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x - y$, $v = 2x + y$ of the triangular region with vertices $(0, 0)$, $(1, 1)$, and $(1, -2)$ in the xy-plane. Sketch the transformed region in the uv-plane.

2. a. Solve the system

$$u = x + 2y, \qquad v = x - y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = x + 2y$, $v = x - y$ of the triangular region in the xy-plane bounded by the lines $y = 0$, $y = x$, and $x + 2y = 2$. Sketch the transformed region in the uv-plane.

3. a. Solve the system

$$u = 3x + 2y, \qquad v = x + 4y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis, and the line $x + y = 1$. Sketch the transformed region in the uv-plane.

4. a. Solve the system

$$u = 2x - 3y, \qquad v = -x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

b. Find the image under the transformation $u = 2x - 3y$, $v = -x + y$ of the parallelogram R in the xy-plane with boundaries $x = -3$, $x = 0$, $y = x$, and $y = x + 1$. Sketch the transformed region in the uv-plane.

Substitutions in Double Integrals

5. Evaluate the integral

$$\int_0^4 \int_{x=y/2}^{x=(y/2)+1} \frac{2x - y}{2}\, dx\, dy$$

from Example 1 directly by integration with respect to x and y to confirm that its value is 2.

6. Use the transformation in Exercise 1 to evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$, and $y = x + 1$.

7. Use the transformation in Exercise 3 to evaluate the integral

$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -(3/2)x + 1$, $y = -(3/2)x + 3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.

8. Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\iint_R 2(x - y)\, dx\, dy.$$

9. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy = 1$, $xy = 9$ and the lines $y = x$, $y = 4x$. Use the transformation $x = u/v$, $y = uv$ with $u > 0$ and $v > 0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}} + \sqrt{xy}\right) dx\, dy$$

as an integral over an appropriate region G in the uv-plane. Then evaluate the uv-integral over G.

10. a. Find the Jacobian of the transformation $x = u$, $y = uv$ and sketch the region G: $1 \le u \le 2$, $1 \le uv \le 2$, in the uv-plane.

b. Then use Equation (1) to transform the integral

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx$$

into an integral over G, and evaluate both integrals.

11. Polar moment of inertia of an elliptical plate A thin plate of constant density covers the region bounded by the ellipse $x^2/a^2 + y^2/b^2 = 1$, $a > 0$, $b > 0$, in the xy-plane. Find the first moment of the plate about the origin. (*Hint:* Use the transformation $x = ar\cos\theta$, $y = br\sin\theta$.)

12. The area of an ellipse The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x, y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk G: $u^2 + v^2 \le 1$ in the uv-plane. Find the area this way.

13. Use the transformation in Exercise 2 to evaluate the integral

$$\int_0^{2/3} \int_y^{2-2y} (x + 2y)e^{(y-x)}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane.

14. Use the transformation $x = u + (1/2)v$, $y = v$ to evaluate the integral

$$\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x - y)e^{(2x-y)^2}\, dx\, dy$$

by first writing it as an integral over a region G in the uv-plane.

15. Use the transformation $x = u/v$, $y = uv$ to evaluate the integral sum

$$\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\, dx\, dy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\, dx\, dy.$$

16. Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\int_0^1 \int_0^{2\sqrt{1-x}} \sqrt{x^2 + y^2}\, dy\, dx.$$

(*Hint*: Show that the image of the triangular region G with vertices (0, 0), (1, 0), (1, 1) in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Finding Jacobians

17. Find the Jacobian $\partial(x, y)/\partial(u, v)$ of the transformation

a. $x = u\cos v$, $y = u\sin v$

b. $x = u\sin v$, $y = u\cos v$.

18. Find the Jacobian $\partial(x, y, z)/\partial(u, v, w)$ of the transformation

a. $x = u\cos v$, $y = u\sin v$, $z = w$

b. $x = 2u - 1$, $y = 3v - 4$, $z = (1/2)(w - 4)$.

19. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is $\rho^2 \sin\phi$.

20. Substitutions in single integrals How can substitutions in single definite integrals be viewed as transformations of regions? What is the Jacobian in such a case? Illustrate with an example.

Substitutions in Triple Integrals

21. Evaluate the integral in Example 5 by integrating with respect to x, y, and z.

22. Volume of an ellipsoid Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of an appropriate region in uvw-space.)

23. Evaluate

$$\iiint |xyz|\, dx\, dy\, dz$$

over the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1.$$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then integrate over an appropriate region in uvw-space.)

24. Let D be the region in xyz-space defined by the inequalities

$$1 \le x \le 2, \quad 0 \le xy \le 2, \quad 0 \le z \le 1.$$

Evaluate

$$\iiint_D (x^2y + 3xyz)\, dx\, dy\, dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw-space.

25. Centroid of a solid semiellipsoid Assuming the result that the centroid of a solid hemisphere lies on the axis of symmetry three-eighths of the way from the base toward the top, show, by transforming the appropriate integrals, that the center of mass of a solid semiellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1,\ z \ge 0$, lies on the z-axis three-eighths of the way from the base toward the top. (You can do this without evaluating any of the integrals.)

26. Cylindrical shells In Section 6.2, we learned how to find the volume of a solid of revolution using the shell method; namely, if the region between the curve $y = f(x)$ and the x-axis from a to b $(0 < a < b)$ is revolved about the y-axis, the volume of the resulting solid is $\int_a^b 2\pi x f(x)\,dx$. Prove that finding volumes by using triple integrals gives the same result. (*Hint:* Use cylindrical coordinates with the roles of y and z changed.)

Chapter 15 Questions to Guide Your Review

1. Define the double integral of a function of two variables over a bounded region in the coordinate plane.
2. How are double integrals evaluated as iterated integrals? Does the order of integration matter? How are the limits of integration determined? Give examples.
3. How are double integrals used to calculate areas and average values. Give examples.
4. How can you change a double integral in rectangular coordinates into a double integral in polar coordinates? Why might it be worthwhile to do so? Give an example.
5. Define the triple integral of a function $f(x, y, z)$ over a bounded region in space.
6. How are triple integrals in rectangular coordinates evaluated? How are the limits of integration determined? Give an example.
7. How are double and triple integrals in rectangular coordinates used to calculate volumes, average values, masses, moments, and centers of mass? Give examples.
8. How are triple integrals defined in cylindrical and spherical coordinates? Why might one prefer working in one of these coordinate systems to working in rectangular coordinates?
9. How are triple integrals in cylindrical and spherical coordinates evaluated? How are the limits of integration found? Give examples.
10. How are substitutions in double integrals pictured as transformations of two-dimensional regions? Give a sample calculation.
11. How are substitutions in triple integrals pictured as transformations of three-dimensional regions? Give a sample calculation.

Chapter 15 Practice Exercises

Evaluating Double Iterated Integrals

In Exercises 1–4, sketch the region of integration and evaluate the double integral.

1. $\displaystyle\int_1^{10}\int_0^{1/y} ye^{xy}\,dx\,dy$
2. $\displaystyle\int_0^1\int_0^{x^3} e^{y/x}\,dy\,dx$
3. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4t^2}}^{\sqrt{9-4t^2}} t\,ds\,dt$
4. $\displaystyle\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xy\,dx\,dy$

In Exercises 5–8, sketch the region of integration and write an equivalent integral with the order of integration reversed. Then evaluate both integrals.

5. $\displaystyle\int_0^4\int_{-\sqrt{4-y}}^{(y-4)/2} dx\,dy$
6. $\displaystyle\int_0^1\int_{x^2}^{x} \sqrt{x}\,dy\,dx$
7. $\displaystyle\int_0^{3/2}\int_{-\sqrt{9-4y^2}}^{\sqrt{9-4y^2}} y\,dx\,dy$
8. $\displaystyle\int_0^2\int_0^{4-x^2} 2x\,dy\,dx$

Evaluate the integrals in Exercises 9–12.

9. $\displaystyle\int_0^1\int_{2y}^{2} 4\cos(x^2)\,dx\,dy$
10. $\displaystyle\int_0^2\int_{y/2}^{1} e^{x^2}\,dx\,dy$
11. $\displaystyle\int_0^8\int_{\sqrt[3]{x}}^{2} \frac{dy\,dx}{y^4+1}$
12. $\displaystyle\int_0^1\int_{\sqrt[3]{y}}^{1} \frac{2\pi\sin\pi x^2}{x^2}\,dx\,dy$

Areas and Volumes Using Double Integrals

13. **Area between line and parabola** Find the area of the region enclosed by the line $y = 2x + 4$ and the parabola $y = 4 - x^2$ in the xy-plane.
14. **Area bounded by lines and parabola** Find the area of the "triangular" region in the xy-plane that is bounded on the right by the parabola $y = x^2$, on the left by the line $x + y = 2$, and above by the line $y = 4$.
15. **Volume of the region under a paraboloid** Find the volume under the paraboloid $z = x^2 + y^2$ above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.
16. **Volume of the region under parabolic cylinder** Find the volume under the parabolic cylinder $z = x^2$ above the region enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the xy-plane.

Average Values

Find the average value of $f(x, y) = xy$ over the regions in Exercises 17 and 18.

17. The square bounded by the lines $x = 1$, $y = 1$ in the first quadrant
18. The quarter circle $x^2 + y^2 \le 1$ in the first quadrant